STABILITY CONDITIONS, WALL-CROSSING AND WEIGHTED GROMOV-WITTEN INVARIANTS

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à Pierre Deligne, en témoignage d'admiration

ABSTRACT. We extend B. Hassett's theory of weighted stable pointed curves ([Has03]) to weighted stable maps. The space of stability conditions is described explicitly, and the wall-crossing phenomenon studied. This can be considered as a non-linear analog of the theory of stability conditions in abelian and triangulated categories (cf. [GKR04], [Bri07], [Joy06, Joy07a, Joy07b, Joy08]).

We introduce virtual fundamental classes and thus obtain weighted Gromov-Witten invariants. We show that by including gravitational descendants, one obtains an $\mathcal{L}$-algebra as introduced in [LM04] as a generalization of a cohomological field theory.

§0. **Introduction: Hassett's stability conditions**

0.1. **Pointed curves.** *A nodal curve* C over an algebraically closed field k is a proper nodal reduced one-dimensional scheme of finite type over this field whose only singularities are nodes. The genus of C is $g := \dim\, H^1(C, \mathcal{O}_C)$.

Let S be a finite set. *A nodal* S*-pointed curve* C is a system $(C, s_i \,|\, i \in S)$ where $\{s_i\}$ is a family of closed non-singular k-points of C, not necessarily pairwise distinct. The element i is called the label of s_i.

The normalization $\widetilde{C}$ of C is a disjoint union of smooth proper curves. Each irreducible component of $\widetilde{C}$ carries inverse images of some labeled points s_i and of singular points of C. Taken together, these points are called *special ones.* Instead of passing to the normalization, we may consider branches (local irreducible germs) of C passing through labeled or singular points. They are in a natural bijection with special points.

A nodal connected S-pointed curve (C, s_i) is called *stable* if $s_i \neq s_j$ for $i \neq j$ and any of the following three equivalent conditions hold:

(i) *The automorphism group of* (C, s_i) *is finite.*

(ii) *Each irreducible component of* $\widetilde{C}$ *of genus 0 (resp. 1) supports* ≥ 3 *(resp.* ≥ 1*) distinct special points.*

(iii) *The line bundle* $\omega_C\left(\sum_{i\in S} s_i\right)$ *is ample.*

This definition has a straightforward extension to families of stable S-pointed curves (cf. below). The basic result states that families of stable S-pointed curves of genus g form (schematic points of) a connected smooth proper over $\mathbb{Z}$ Deligne-Mumford stack $\overline{\mathcal{M}}_{g,S}$. It contains an open dense substack $\mathcal{M}_{g,S}$ parameterizing irreducible smooth curves, and is its compactification.

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0.2. **Weighted stability.** Generalizing condition (iii), B. Hassett enriched the theory by additional parameters generating a whole new family of stability conditions, which lead to new moduli stacks, representing different compactifications of $\mathcal{M}_{g,S}$.

Namely, *the weight data* on S is a function $\mathcal{A}: S \to \mathbb{Q}$, $0 < \mathcal{A}(i) \leq 1$. S together with a weight data will be called *a weighted set.*

0.2.1. **Definition** ([Has03]). *A connected S-pointed curve $(C, s_i \,|\, i \in S)$ is called weighted stable (with respect to $\mathcal{A}$) if the following conditions are satisfied:*

(i) $K_C + \sum_i \mathcal{A}(i) s_i$ is an ample divisor, where K_C is the canonical class of C.

(ii) For any subset $I \subset S$ such that s_i pairwise coincide for $i \in I$, we have $\sum_{i \in I} \mathcal{A}(i) \leq 1$.

Clearly, (i) implies that $2g - 2 + \sum_i \mathcal{A}(i) > 0$.

The usual stability notion corresponds to the case $\mathcal{A}(i) = 1$ for all $i \in S$. Independently of Hassett's work, A. Losev and Yu. Manin considered in [LM00], [LM04] some non-standard moduli spaces which turned out to correspond to special Hassett's stability conditions: see [Has03, section 6.4] and [Man04].

Definition 0.2.1 admits a straightforward extension to families:

Let U be a scheme, S a finite set, $g \geq 0$. *An S-pointed nodal curve (or family of curves) of genus g over U* consists of the data

$$(\pi: C \to U;\, s_i: U \to C,\ i \in S)$$

where π is a flat proper morphism whose geometric fibers C_t are nodal S-pointed curves of genus g.

This family is called $\mathcal{A}$-stable iff

(i) $K_\pi + \sum_i \mathcal{A}(i) s_i$ *is π-relatively ample.*

(ii) *For any $I \subset S$ such that $\cap_{i \in I} s_i \neq \emptyset$, we have* $\sum_{i \in I} \mathcal{A}(i) \leq 1$.

0.3. **Stacks of weighted stable curves $\overline{\mathcal{M}}_{g,\mathcal{A}}$.** The first main result of [Has03] is a proof of the following fact. Fix a weighted set of labels S and a value of genus g. Then families of weighted stable S-pointed curves of genus g form (schematic points of) a connected smooth proper over $\mathbb{Z}$ Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$. The respective coarse moduli scheme is projective over $\mathbb{Z}$.

0.4. **Walls and wall-crossing.** The further results of Hassett on which we focus in this introduction concern the geometry of the space of stability conditions governing the varying geometry of boundaries of $\overline{\mathcal{M}}_{g,\mathcal{A}}$ ([Has03], sec. 5).

Put

$$D_{g,S} := \{\mathcal{A} \in \mathbb{R}^S \,|\, 0 < \mathcal{A}(i) \leq 1,\ \sum_s \mathcal{A}(i) > 2 - 2g\}.$$

Walls are non-empty intersections of $D_{g,n}$ with certain hyperplanes indexed by subsets $I \subset S$:

$$w_I := \{\mathcal{A} \in D_{g,S} \,|\, \sum_{i \in I} \mathcal{A}(i) = 1\,\}.$$

Coarse chambers are defined as connected components of

$$D_{g,S} - \bigcup_{2 < |I| \leq n - 3\delta_{g,0}} w_I.$$

Fine chambers are connected components of

$$D_{g,S} - \bigcup_{2\leq|I|\leq n-2\delta_{g,0}} w_I.$$

B. Hassett proves the following result:

0.4.1. **Proposition.** *(i) The moduli stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is constant on each coarse chamber, and differs from one coarse chamber to another.*

(ii) The universal curve $\overline{\mathcal{C}}_{g,\mathcal{A}}$ is constant on each fine chamber, and differs from one fine chamber to another.

Finally, for any point $\mathcal{A}'$ belonging to a wall, there exists a point $\mathcal{A}$ inside a neighboring coarse (resp. fine) chamber at which $\overline{\mathcal{M}}_{g,\mathcal{A}}$ (resp. $\overline{\mathcal{C}}_{g,\mathcal{A}}$) is the same as at $\mathcal{A}'$.

0.5. **Plan of this paper.** Let V be a smooth projective manifold. M. Kontsevich has defined stacks $\mathcal{M}_{g,S}(V)$ of S-pointed stable maps $(C \to V; s_i)$. The stacks $\mathcal{M}_{g,S}$ correspond to the case V = a point. In this paper we generalize Hassett's stability conditions to $\mathcal{M}_{g,S}(V)$ and study the resulting stacks.

In §1, we define the precise moduli problem and construct its moduli space as a proper Deligne-Mumford stack. We show the existence of birational contraction morphism for any reduction of the weights; in particular, all moduli spaces of weighted stable maps are birational contractions of the Kontsevich moduli space.

We establish the existence of all basic morphisms (gluing, changing the target, forgetting sections etc.) between them in §2. Section §3 describes the chamber decompositions of the set of admissible weights. and exhibits the reduction morphisms for a wall-crossing as an explicit blow-up.

In §4, we postulate a list of basic properties for virtual fundamental classes, and discuss consequences for the weighted Gromov-Witten invariants. After introducing the language of weighted graphs in §5, we prove a more complete graph-level list of properties of the virtual fundamental classes in §6.

One motivation of this study was the work by Losev and Manin on painted stable curves [LM00, LM04, Man04], which constitute a special case of weighted stable curves. The authors introduced the notion of an $\mathcal{L}$-algebra as an extension of the notion of a cohomological field theory of [KM94].

The construction of virtual fundamental classes in the extended context of new stability conditions allows us to produce Gromov-Witten invariants based on weighted stable maps. Including gravitational descendants, we obtain $\mathcal{L}$-algebras in the sense of [LM04]. While weighted Gromov-Witten invariants without gravitational descendants yield nothing new (see proposition 4.2.1), the coupling to gravity in the weighted case exhibits a new structure on quantum cohomology.

In [MM08], the authors already constructed moduli spaces of weighted stable maps as an aide in computing the Chow ring of non-weighted stable maps with target $\mathbb{P}^n$. Independently of the present paper, Alexeev and Guy studied the behavior of gravitational descendants for changes of weights in [AG08], assuming the same definition of virtual fundamental classes that we study in sections §4 and §6.

Another motivation is spelled down below.

0.6. **Stability conditions in abelian and triangulated categories.** Stability conditions have been generally designed to choose a *preferred* compactification of various

moduli spaces, typically of vector bundles, or more general coherent sheaves on projective manifolds. It was only recently that the attention of algebraic geometers shifted to the families of variable stability conditions and their geometry: see [GKR04], [Bri07], [Joy06, Joy07a, Joy07b, Joy08], and the references therein. An influential recent paper by T. Bridgeland [Bri07] was very much stimulated by physics work on mirror symmetry, in particular, M. Douglas's notion of Π-stability.

In this subsection we will sketch a purely geometric context in which various notions of stability in derived categories of coherent sheaves might be quite useful (see [Ina02], [Bri02], for a version of background notions).

Namely, consider the problem treated in several papers by A. Bondal, D. Orlov and others: *what can be said about a (smooth projective) manifold V if we know its bounded derived category of coherent sheaves $D(V)$?*

In an important paper [BO01] it was shown that if the canonical sheaf Ω_V of V or its inverse is ample, then V can be reconstructed up to an isomorphism from $D(V)$. The strategy of proof is this: the authors show how to detect (up to a shift) classes of structure sheaves of closed points of V in $D(V)$, then classes of invertible sheaves, and finally to reconstruct the canonical (or anticanonical) homogeneous coordinate ring.

This result can become dramatically wrong, when $\Omega_V^{\pm 1}$ is not ample, for example, when it is trivial. In the proper Calabi-Yau case various birational models may lead to equivalent derived categories. The complete picture in this case is far from being clear. The proof that worked in the Fano/anti-Fano cases breaks down at the first step: the classes of structure sheaves of closed points of V become unrecognizable.

However, the general strategy of the proof could be saved without additional assumptions on Ω_V if one could do the following:

a) Devise a family of appropriate stability conditions $\mathcal{C}$ (this is probably already done in [Bri07]).

b) Prove that various V's with "the same" $D(V)$ could be reconstructed as moduli spaces $V_{\mathcal{C}}$ of appropriately defined $\mathcal{C}$-stable point-like complexes in $D(V)$. The deformation theory of objects in derived categories is not yet a mature subject, but see [LO06] for some recent developments.

c) obtain a sufficiently detailed description of chamber decomposition and wall-crossing in the space of $\mathcal{C}$'s.

A tentative picture of this type can be glimpsed from the Aspinwall's sketch [Asp03]. Locally, the wall-crossing phenomenon has been studied in [Tod08].

Notice however that it is not clear a priori what would be the net outcome of such a reasoning. In fact, according to the recent preprint [Căl07], two Calabi-Yau threefolds can have equivalent derived categories without being birationally equivalent.

On the positive side, however, they must have isomorphic motives: cf. [Orl05].

From this perspective, Hassett's theory and its generalization, discussed in this paper, can be perceived as a toy model for the more sophisticated case of the triangulated categories. Moreover, various notions of stability for maps of curves into nontrivial target spaces could conceivably be combined with similar stability notions for complexes of sheaves on the target space leading to a richer structure of quantum cohomology.

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§1. **Geometry of moduli spaces of weighted stable maps**

1.1. **The moduli problem.** Let k be a field of any characteristic, V/k a projective variety, and $\beta \in \mathrm{CH}^1(V)$ an effective one-dimensional class in the Chow ring. Let S be a finite set with weights $\mathcal{A}\colon S \to \mathbb{Q} \cap [0,1]$, and let $g \geq 0$ be any genus.

1.1.1. **Definition.** *A nodal curve of genus g over a scheme T/k is a proper, flat morphism $\pi\colon C \to T$ of finite type such that for every geometric point* $\operatorname{Spec} \eta$ *of T, the fiber over* $\operatorname{Spec} \eta$ *is a connected curve of genus g with only nodes as singularities.*

Given $(g, S, \mathcal{A}, \beta)$ as above, a prestable map of type $(g, \mathcal{A}, \beta)$ over T is a tuple (C, π, s, f) where $\pi\colon C \to T$ is a nodal curve of genus g, $s = (s_i)_{i \in S}$ is an S-tuple of sections $s_i\colon T \to C$, and f is a map $f\colon C \to V$ with $f_*([C]) = \beta$, such that

(1) *the image of any section s_i with positive weight $\mathcal{A}(i) > 0$ lies in the smooth locus of C/T,*
(2) *for any subset $I \subset S$ such that the intersection $\bigcap_{i \in I} s_i(T)$ of the corresponding sections is non-empty, we have $\sum_{i \in I} \mathcal{A}(i) \leq 1$.*

1.1.2. **Definition.** *A stable map of type $(g, \mathcal{A}, \beta)$ over T is a prestable map (C, π, s, f) of the same type such that $K_\pi + \sum_{i \in S} \mathcal{A}(i) s_i + 3 f^*(M)$ is π-relatively ample for some ample divisor M on V.*

We will often call such a curve $(g, \mathcal{A})$-stable when the homology class β is irrelevant.

1.1.3. **Remark.** *Assume that (C, π, s, f) is a $(g, \mathcal{A})$-prestable map over T. Then it is $(g, \mathcal{A})$-stable if and only if it is $(g, \mathcal{A})$-stable over geometric points of T.*

Over an algebraically closed field, ampleness of $K_\pi + \sum_{i \in S} \mathcal{A}(i) s_i + 3 f^*(M)$ can only fail on irreducible components C that are of genus 0 and get mapped to a point by f. Precisely, if n_C is the number of inverse images of nodal points in the normalization, then ampleness is equivalent to $n_C + \sum_{i\colon s_i \in C} \mathcal{A}(i) > 2$.

In particular, stability can be checked with an arbitrary ample divisor M; if all sections have weight 1 (we will write this as $\mathcal{A} = \underline{1}_S$), weighted stability agrees with the definition of a stable map by Kontsevich.

We call the data $g, S, \mathcal{A}, \beta$ *admissible*, if $\beta \neq 0$ or $2g - 2 + \sum_{i \in S} \mathcal{A}(i) > 0$, and if β is bounded by the characteristic (cf. [BM96, Theorem 3.14]: this means that k has characteristic zero, or that $\beta \cdot L < \operatorname{char} k$ for some very ample line bundle L on V).

1.1.4. **Theorem.** *Given admissible data $g, S, \mathcal{A}, \beta$, let $\overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$ be the category of stable maps of type $(g, \mathcal{A}, \beta)$ and their isomorphisms, with the standard structure as a groupoid over schemes over* $\operatorname{Spec} k$*.*

This category is a proper algebraic Deligne-Mumford stack of finite type.

The property of being a stack follows from standard arguments. The geometric properties are proven in section 1.3. Some of their proofs are simplified by the use of the contraction morphism from the Kontsevich moduli space $\overline{\mathcal{M}}_{g,S}(V, \beta)$ to the space of weighted stable maps as discussed in the next section; hence their existence will be proved first.

1.2. **Reduction morphisms for weight changes.** If $\beta \neq 0$, consider the open and dense substack

$$C_{g,S}(V, \beta) \subset \overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$$

of maps that do not contract any irreducible component of genus zero, and for which all marked sections are distinct. By some abuse of language we will call $C_{g,S}(V,\beta)$ the "configuration space". Since any such map is stable regardless of the choice of weights, $C_{g,S}(V,\beta)$ does not depend on $\mathcal{A}$. Every $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ is a compactification of $C_{g,S}(V,\beta)$, and thus all the moduli stacks for different $\mathcal{A}$ are birational. The following proposition gives actual morphisms, provided that the weights are comparable. They will be analyzed in more detail in §3.

Consider two weights $\mathcal{A},\mathcal{B}\colon S\to\mathbb{Q}\cap[0,1]$ such that $\mathcal{A}(i)\geq\mathcal{B}(i)$ for all $i\in S$; we will just write $\mathcal{A}\geq\mathcal{B}$ from now on. Any $(g,\mathcal{A})$-stable map is obviously $(g,\mathcal{B})$-prestable, but it may not be $(g,\mathcal{B})$-stable. However, we can stabilize the curve with respect to $\mathcal{B}$:

1.2.1. **Proposition.** *If $g,S,\beta,\mathcal{A}\geq\mathcal{B}$ are as above, there is a natural reduction morphism*

$$\rho_{\mathcal{B},\mathcal{A}}\colon\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)\to\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta).$$

It is surjective and birational.[1] *Over an algebraically closed field η, it is given by adjusting the weights and then successively contracting all $(g,\mathcal{B})$-unstable components.*

Given three weight data $\mathcal{A}\geq\mathcal{B}\geq\mathcal{C}$, the reduction morphisms respect composition: $\rho_{\mathcal{C},\mathcal{A}}=\rho_{\mathcal{C},\mathcal{B}}\circ\rho_{\mathcal{B},\mathcal{A}}$.

In particular, every moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ is a birational contraction of the Kontsevich moduli space $\overline{\mathcal{M}}_{g,S}(V,\beta)=\overline{\mathcal{M}}_{g,\underline{1}_S}(V,\beta)$.

1.3. **Proofs of the geometric properties.** As in the case of $(g,\mathcal{A})$-stable curves, the following vanishing result is essential to ensure that all constructions are compatible with base change:

1.3.1. **Proposition.** [Has03, Proposition 3.3] *Let C be a connected nodal curve of genus g over an algebraically closed field, D an effective divisor supported in the smooth locus of C, and L an invertible sheaf with $L\cong\omega_C^k(D)$ for $k>0$.*

1. If L is nef, and $L\neq\omega_C$, then L has vanishing higher cohomology.

2. If L is nef and has positive degree, then L^N is basepoint free for $N\geq 2$.

3. If L is ample, then L^N is very ample when $N\geq 3$.

4. Assume L is nef and has positive degree, and let C' denote the image of C under L^N with $N\geq 3$. Then C' is a nodal curve with the same arithmetic genus as C, obtained by collapsing the irreducible components of C on which L has degree zero. Components on which L has positive degree are mapped birationally onto their images.

1.3.2. **Stability and geometric points.** We will first show how remark 1.1.3 follows from this proposition: Consider the line bundle

$$L=\omega_C^k(k\sum_{i\in S}\mathcal{A}(i)s_i)\otimes f^*(\mathcal{O}(M))^{3k},$$

where k is such that all numbers $k\mathcal{A}(i)$ are integral. Then by the proposition and the base change theorems, formation of $P:=\mathrm{Proj}(\pi_*(L^N))$ commutes with base change.

[1] It is an isomorphism over the open subset $U:=C_{g,S}(V,\beta)$, which satisfies the following strong density property: for any open subset V, there is no non-zero section $f\in\mathcal{O}(V)$ that vanishes on $U\cap V$; in other words, the complement is nowhere dense and does not have *additional* nilpotent structure.

By definition, L is relatively ample iff the induced morphism $p\colon C \to P$ is defined everywhere and an open immersion. By [SGA1, exposé I, Théorème 5.1], this is the case if and only if p is everywhere defined, radical, flat and unramified. All these conditions can be checked on geometric fibers (for flatness, this follows from [EGA, IV, Théorème 11.3.10], for unramifiedness from the conormal sequence).

1.3.3. **Reduction morphisms.** By Grothendieck's descent theory, $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ is a stack in the étale topology, i. e. the Isom functors are sheaves and any étale descent datum is effective. We first show the existence of the natural reduction morphisms $\rho_{\mathcal{B},\mathcal{A}}$ as maps between these abstract stacks. This will enable us to use the results of [BM96] on $\overline{\mathcal{M}}_{g,S}(V,\beta)$ to shorten our proofs.

Using the vanishing result 1.3.1, the proof of proposition 1.2.1 is completely analogous to that of theorem 4.1 in [Has03]: Let $\mathcal{B}_\lambda = \lambda\mathcal{A} + (1-\lambda)\mathcal{B}$, and let $1 = \lambda_0 > \lambda_1 > \cdots > \lambda_N = 0$ be a finite set such that for all $\lambda \notin \{\lambda_0, \ldots, \lambda_N\}$, the following condition holds:

- *There is no subset $I \subset S$ such that $\sum_{i\in I} \mathcal{B}_\lambda(i) = 1$ and $\sum_{i \in I} \mathcal{B}_1(i) \neq 1$.* (*)

We will construct $\rho_{\mathcal{B},\mathcal{A}}$ as the composition $\rho_{\mathcal{B},\mathcal{A}} = \rho_{\mathcal{B}(\lambda_N),\mathcal{B}(\lambda_{N-1})} \circ \cdots \circ \rho_{\mathcal{B}(\lambda_1),\mathcal{B}(\lambda_0)}$. This means we can assume that the condition (*) holds for all $0 < \lambda < 1$.

Fix an ample divisor M on V, and fix a natural number k so that $k\mathcal{B}(i)$ is an integer for all i. Let L be the invertible sheaf $L := \omega_C^k(k\sum_{i\in S}\mathcal{B}(i)s_i) \otimes f^*(M)^3$ for any $(g,\mathcal{A})$-stable map $f\colon C \to V$ over T. Due to condition (*), it is nef; also it has positive degree. Let C' be the image of C under the map induced by L^N for some $N \geq 3$, i.e. $C' = \operatorname{Proj} \mathcal{R}$ where $\mathcal{R}$ is the graded sheaf of rings on T given by $\mathcal{R}_l = \pi_*((L^N)^l)$. Let $t\colon C \to C'$ be the natural map, and let $s_i' = t \circ s_i$. By the same arguments as in the non-weighted case, C' is a nodal curve of genus g, and s_i' lie in the smooth locus whenever $\mathcal{B}(i) > 0$. By proposition 1.3.1, L has vanishing higher cohomology; so the formation of $\pi_*((L^N)^l)$ and hence that of C' commutes with base change. Over an algebraically closed field, this morphism agrees with the description via contraction of unstable components. In particular, C' is $(b,\mathcal{B})$-prestable.

The original f factors via the induced morphism $f'\colon C' \to V$. Let L' be the line bundle $L' := \omega_{C'}^k(k\sum_{i\in S}\mathcal{B}(i)s_i) \otimes f'^*(M)^3$. Then $t_*L = L'$; hence L' is ample and (C', π', s', f') is a $(g,\mathcal{B})$-stable map. The induced morphism $T \to \overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$ commutes with base change and thus yields the map $\rho_{\mathcal{B},\mathcal{A}}$ between stacks as claimed.

To prove surjectivity, it is sufficient to show that every $(g,\mathcal{B})$-stable map (C,s,f) over an algebraically closed field K is the image of some $(g,\mathcal{A})$-stable map (C',s',f') over K. It is obvious how to construct C': If $I \subset S$ is a subset of the labels such that condition (2) of definition 1.1.1 is violated for the weight data $\mathcal{A}$, i.e. the marked points $s_i, i \in I$ coincide and $\sum_{i\in I}\mathcal{A}(i) > 1$, we can attach a copy of $\mathbb{P}^1(K)$ at this point, move the marked points to arbitrary but different points on $\mathbb{P}^1$, and extend the map constantly along $\mathbb{P}^1$.

Birationality (for $\beta \neq 0$) follows from the fact that $\rho_{\mathcal{B},\mathcal{A}}$ is an isomorphism over the configuration space $C_{g,S}(V,\beta)$, which is a dense and open subset. The compatibility with composition follows immediately once we have shown the the moduli spaces are separate: the two morphisms $\rho_{\mathcal{C},\mathcal{A}}$ and $\rho_{\mathcal{C},\mathcal{B}} \circ \rho_{\mathcal{B},\mathcal{A}}$ agree on the configuration space.

1.3.4. **Proposition.** *The diagonal $\Delta\colon \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta) \times \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ is representable, separated and finite.*

Let (C_1, π_1, s_1, f_1) and (C_2, π_2, s_2, f_2) be two families of $(g, \mathcal{A})$-stable maps to V over a scheme T. We have to show that $\underline{\mathrm{Isom}}((C_1, \pi_1, s_1, f_1), (C_2, \pi_2, s_2, f_2))$ is represented by a scheme finite and separated over T. Since V is projective and β is bounded by the characteristic, we can use exactly the same argument as in the proof of [BM96, Lemma 4.2]: one shows that étale locally on T, one can extend the set of labels to $S \cup S'$ and find additional S'-tuples of sections $(s_1)'$ and $(s_2)'$, such that $(C_1, \pi_1, s_1 \cup s_1')$ and $(C_2, \pi_2, s_2 \cup s_2')$ are $(g, \mathcal{A} \cup \underline{1}_{S'})$-stable curves, and that there is a natural closed immersion

$$\underline{\mathrm{Isom}}((C_1, \pi_1, s_1, f_1), (C_2, \pi_2, s_2, f_2)) \to \underline{\mathrm{Isom}}((C_1, \pi_1, (s_1, s_1')), (C_2, \pi_2, (s_2, s_2'))).$$

Sine $\overline{\mathcal{M}}_{g, \mathcal{A} \cup \underline{1}_{S'}}$ has a representable, separated and finite diagonal by [Has03], the claim of the proposition follows.

1.3.5. **Existence as Deligne-Mumford stacks.** In particular, the diagonal is proper and thus the moduli stack separated. As $\overline{\mathcal{M}}_{g, \underline{1}_S}(V, \beta)$ is proper and the reduction morphism $\rho_{\mathcal{A}, \underline{1}_S} : \overline{\mathcal{M}}_{g,S}(V, \beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$ is surjective, $\overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$ is also proper.

Finally, the existence of a flat covering of finite type follows with almost the same argument as the one in [BM96], following Proposition 4.7 there. However, some changes are required, so we spell it out in detail: We write $\mathcal{A}_n = \mathcal{A} \cup \underline{1}_{\{1,\dots,n\}}$ for the weight data obtained from $\mathcal{A}$ by adding n weights of 1. Let $\overline{\mathcal{M}}^o_{g,\mathcal{A}_n}(V, \beta)$ be the open substack of $\overline{\mathcal{M}}_{g,\mathcal{A}_n}(V, \beta)$ where the additional sections of weight one lie in the smooth locus of $C_{g,\mathcal{A}}(V, \beta)$ and away from the existing sections (in other words, the open substack where the map is already $(g, \mathcal{A})$-stable after forgetting the additional sections). The obvious forgetful map

$$\phi^0_{\mathcal{A},\mathcal{A}_n} : \overline{\mathcal{M}}^o_{g,\mathcal{A}_n}(V, \beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$$

is smooth and in particular flat. Let $U^0_{g,\mathcal{A}_n}(V, \beta)$ be the open substack of $\overline{\mathcal{M}}^o_{g,\mathcal{A}_n}(V, \beta)$ where the curve is already $(g, \mathcal{A}_n)$-stable as a curve. Then for high enough n, the restriction $\phi^o_{\mathcal{A},\mathcal{A}_n}|_{U^0_{g,\mathcal{A}_n}(V,\beta)}$ to this substack is surjective. On the other hand, $U^0_{g,\mathcal{A}_n}(V, \beta)$ is an open substack of the relative morphism space $\mathrm{Mor}_{\overline{\mathcal{M}}_{g,\mathcal{A}_n}}(V, \beta)$ (parameterizing maps $T \to \overline{\mathcal{M}}_{g,\mathcal{A}_n}$ together with a map of the pull-back of the universal curve $C_{g,\mathcal{A}_n}$ to V). So a flat presentation of the morphism space induces one for $\overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$.

§2. **Elementary morphisms**

2.1. **Gluing morphisms.** As in the non-weighted case, we can glue curves at marked points, but to guarantee that the resulting curves are prestable, we have to assume that both labels have weight 1:

Let $g_1, S_1, \mathcal{A}_1, \beta_1$ and $g_2, S_2, \mathcal{A}_2, \beta_2$ be weight data, such that the extensions $g_i, S_i \cup \{0\}, \mathcal{A}_i \cup \{0 \mapsto 1\}, \beta_i$ by an additional label of weight 1 are admissible. Denote by ev_0 be the evaluation morphisms $\mathrm{ev}_0 : \overline{\mathcal{M}}_{g_i,\mathcal{A}_i \cup \{1\}}(V, \beta_i) \to V$ given by evaluating the additional section: $\mathrm{ev}_0 = f \circ s_0$. Similarly, let $g, S, \mathcal{A}, \beta$ be weight data such that $g, S \cup \{0, 1\}, \mathcal{A} \cup \{1, 1\}, \beta$ is admissible, and let $\mathrm{ev}_0, \mathrm{ev}_1$ be the additional evaluation morphisms.

2.1.1. **Proposition.** *There are natural gluing morphisms*

$$\big(\overline{\mathcal{M}}_{g_1,\mathcal{A}_1\cup\{1\}}(V,\beta_1)\times\overline{\mathcal{M}}_{g_2,\mathcal{A}_2\cup\{1\}}(V,\beta_2)\big)\times_{V\times V}V\to\overline{\mathcal{M}}_{g_1+g_2,\mathcal{A}_1\cup\mathcal{A}_2}(V,\beta_1+\beta_2)$$

and

$$\overline{\mathcal{M}}_{g,\mathcal{A}\cup\{1,1\}}(V,\beta)\times_{V\times V}V\to\overline{\mathcal{M}}_{g+1,\mathcal{A}}(V,\beta).$$

The product over $V\times V$ is taken via the morphism $(\mathrm{ev}_0,\mathrm{ev}_0)$ respectively $(\mathrm{ev}_0,\mathrm{ev}_1)$ on the left, and the diagonal $\Delta\colon V\to V\times V$ on the right.

There is nothing new to prove here, except to note that the weight of 1 guarantees that the marked sections (of positive weight) do not meet the additional node on the glued curve.

2.2. **Proposition.** *Let $\mu\colon V\to W$ be a morphism, and $(g,S,\mathcal{A},\beta)$ be admissible data for V, such that $(g,S,\mathcal{A},\mu_*(\beta))$ is also admissible. Then there is a natural push-forward*

$$\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)\to\overline{\mathcal{M}}_{g,\mathcal{A}}(W,\mu_*(\beta))$$

that is obtained by composing the maps with μ, followed by stabilization.

One could adapt the proof of [BM96] to the weighted case; instead, we give a proof analogous to the one in section 1.3.3.

Let $f\colon C\to V$ be the universal map over $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$, let $f'=\mu\circ f$ be the induced map to W, and let M' be an ample divisor on V'. By the assumptions, the divisor $D'=K_\pi+\sum_{i\in S}\mathcal{A}(i)s_i+3f'^*M'$ has positive degree; however, it need not be nef. Hence we consider $D=K_\pi+\sum_{i\in S}\mathcal{A}(i)s_i+3f^*M$ and $D(\lambda)=\lambda D+(1-\lambda)D'$ for $0\le\lambda\le 1$. Let $\{\lambda_1,\dots,\lambda_N\}$ be the set of λ for which the degree of $D(\lambda)$ is zero on any irreducible component of C, and let $k_r, r=1\dots N$ be an integer such that $k_r\lambda_r$ and $k_r\mathcal{A}(i), i\in S$ is integer.

Then $L_1=\omega^{k_1}(k_1\sum_{i\in S}\mathcal{A}(i)s_i+k_1(3f^*M\lambda_1+(1-\lambda_1)3f'^*M'))$ is a nef invertible sheaf on C for which proposition 1.3.1 applies. Hence C_1 defined by $C_1:=\operatorname{Proj}\mathcal{R}_1$ and $(\mathcal{R}_1)_l=\pi_*(L_1^{3l})$ is again a flat nodal curve of genus g, contracting all components of C on which L_1 fails to be ample, and f' factors via a unique morphism $f_1\colon C_1\to W$. We proceed inductively to obtain $f_N\colon C_N\to W$ on which D' is ample; this induces the map of moduli stacks. Note that $C\to C_N\to W$ is the universal factorization of f' such that $f_N\colon C_N\to W$ is a $(g,\mathcal{A})$-stable map.

2.3. **Proposition.** *Given admissible weight data $(g,S,\mathcal{A},\beta)$, let $(g,S\cup\{*\},\mathcal{A}\cup\{a\}=\mathcal{A}\coprod\{*\mapsto a\},\beta)$ be the weight data obtained by adding a label $\{*\}$ of arbitrary weight $a\in\mathbb{Q}\cap[0,1]$. There is a natural forgetful map*

$$\phi_{\mathcal{A},\mathcal{A}\cup\{a\}}\colon\overline{\mathcal{M}}_{g,\mathcal{A}\cup\{a\}}(V,\beta)\to\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$$

obtained by forgetting the additional section and stabilization. If $a=0$, then $\phi_{A,A\cup\{0\}}$ is the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$.

We can construct this map as the composition

$$\phi_{\mathcal{A},\mathcal{A}\cup\{0\}}\circ\rho_{\mathcal{A}\cup\{0\},\mathcal{A}\cup\{a\}}\colon\overline{\mathcal{M}}_{g,\mathcal{A}\cup\{a\}}(V,\beta)\to\overline{\mathcal{M}}_{g,\mathcal{A}\cup\{0\}}(V,\beta)\to\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta).$$

The second morphism $\phi_{\mathcal{A},\mathcal{A}\cup\{0\}}$ is the naive forgetful morphism, as a map is $(g,\mathcal{A}\cup\{0\})$-stable if and only if it is $(g,\mathcal{A})$-stable.

2.4. **Proposition.** *Let $S' \coprod S'' = S$ be a partition of the set of labels such that $\mathcal{A}(S'') = \sum_{i \in S''} \mathcal{A}(i) \leq 1$. Then there is a natural map*

$$\overline{\mathcal{M}}_{g,\mathcal{A}|_{S'} \cup \{\mathcal{A}(S'')\}}(V, \beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta).$$

It is given by setting $s_i = s_*$ for all $i \in S'$. It identifies $\overline{\mathcal{M}}_{g,\mathcal{A}|_{S'} \cup \{* \mapsto \mathcal{A}(S'')\}}(V, \beta)$ with the locus of $\overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$ where all $s_i, i \in S''$ agree.

2.5. **Weighted marked graphs.** A graph was defined in [BM96] as a quadruple $\tau = (V_\tau, F_\tau, \partial_\tau, j_\tau)$ of a set of vertices V_τ, a set of flags F_τ, a morphism $\partial_\tau \colon F_\tau \to V_\tau$ and an involution $j_\tau \colon F_\tau \to F_\tau$. We think of a graph in terms of its geometric realization: it is obtained by identifying in the disjoint union $\coprod_{f \in F_\tau} [0, 1]$ the points 0 for all flags f attached to the same vertex via $v = \partial_\tau(f)$, and the points 1 for all orbits of j_τ. A flag f with $j_\tau(f) = f$ is called a *tail* of the vertex $\partial_\tau(f)$, whereas a pair $\{f, j_\tau(f)\}$ for $f \neq j_\tau(f)$ is called an *edge*, connecting the (not necessarily distinct) vertices $\partial_\tau(f)$ and $\partial_\tau(j_\tau(f))$.

Given a projective variety V, a weighted modular V-graph is a graph τ together with a genus $g \colon V_\tau \to \mathbb{Z}_{\geq 0}$, a weight data $\mathcal{A} \colon F_\tau \to \mathbb{Q} \cap [0, 1]$ such that $\mathcal{A}(f) = 1$ for all flags that are part of an edge, and a marking $\beta \colon V_\tau \to H_2^+(V)$. To any weighted stable map we can associate its dual graph: a vertex for every irreducible component, an edge for every node, and a tail for every marked section. Conversely, to every weighted modular graph we can associate the moduli space of tuples of weighted stable maps $f_v \colon C_v \to V$ of type $(g(v), S_v = \{f \in F_\tau \colon \partial(f) = v\}, \mathcal{A}|_{S_v}, \beta(v))$, such that for every edge $\{f, f' = j_\tau(f)\}$ connecting the vertices $v = \partial_\tau(f)$ and $v' = \partial_\tau(f)$, the corresponding evaluation morphisms are identical: $f_v \circ s_f = f_{v'} \circ s_{f'}$. Via gluing, this gives a single weighted stable map $f \colon C \to V$; if all C_v are smooth, its dual graph will give back τ.

The moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$ corresponds to the one-vertex graphs with the set S of tails. The morphisms constructed in this section correspond to elementary morphisms between graphs with one and two vertices. Extending this set of morphisms to higher codimension boundary strata, indexed by graphs with more vertices, naturally leads to a category of weighted stable marked graphs. We will adopt this viewpoint in §5, and show that we get a functor $\overline{\mathcal{M}}$ from the graph category to Deligne-Mumford stacks over k.

§3. Birational behavior under weight changes

For this section, we will fix g, S, V, β, and analyze more systematically the reduction morphisms $\rho_{\mathcal{A},\mathcal{B}}$ of proposition 1.2.1 for varying weight data $\mathcal{A}, \mathcal{B}$. Assume that g, V, β are such that $\overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$ is not empty.

3.1. Exceptional locus and reduction morphism as blow-up.

3.1.1. **Proposition.** [Has03, Proposition 4.5] *Assume we have weight data $\mathcal{A} \geq \mathcal{B} > 0$. The reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$ contracts the boundary divisors $D_{I,J}$ given as the image of the gluing morphism*

$$\overline{\mathcal{M}}_{0,\mathcal{A}|_I \cup \{1\}}(V, 0) \times_V \overline{\mathcal{M}}_{g,\mathcal{A}|_J \cup \{1\}}(V, \beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V, \beta)$$

for all partitions $I \coprod J = S$ of S with

$$\sum_{i \in I} \mathcal{A}(i) > 1 \quad \textit{and} \quad b_I := \sum_{i \in I} \mathcal{B}(i) \leq 1.$$

There is a factorization of $\rho_{\mathcal{B},\mathcal{A}}|D_{I,J}$ via

$$\overline{\mathcal{M}}_{0,\mathcal{A}|_I \cup 1}(V,0) \times_V \overline{\mathcal{M}}_{g,\mathcal{A}|_J \cup 1}(V,\beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}|_J \cup \{1\}}(V,\beta) \to \overline{\mathcal{M}}_{g,\mathcal{A}|_J \cup \{b_I\}}(V,\beta).$$

We may assume that there is just one such I and that $b_I = 1$. The stabilization contracts components on which $\omega_C^k(k \sum_{i \in S} \mathcal{B}(i) s_i) \otimes f^*(M)^3$ has degree zero. Such a component can only be a smooth irreducible component of genus zero that is mapped to a point, meets the other components in a single node and contains exactly those marked sections s_i with $i \in I$.

In particular, the exceptional locus of $\rho_{\mathcal{B},\mathcal{A}}$ is given by all $D_{I,J}$ for partitions $I \cap J = S$ as above with the additional condition $|I| > 2$. When all sets $I \subset S$ such that $\sum_{i \in I} \mathcal{A}(i) > 1$ and $\sum_{i \in I} \mathcal{B}(i) \leq 1$ satisfy $|I| = 2$, then $\rho_{\mathcal{B},\mathcal{A}}$ is an isomorphism.

3.1.2. **Remark.** *Assume that for $\mathcal{A} > \mathcal{B} > 0$, there is exactly one partition $I \coprod J = S$ of S as in the proposition. Then $\rho_{\mathcal{B},\mathcal{A}}$ is the blow-up of $\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$ along the substack $C_{IJ} \cong \overline{\mathcal{M}}_{g,\mathcal{B}|_J \cup \{b_I\}}(V,\beta)$ of weighted stable curves where all section s_i for $i \in I$ are identical.*

We first show that there is a natural map from $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ to the blow-up: The divisor $D_{I,J}$ is the scheme-theoretic inverse image of C_{IJ}. Further, it is a Cartier divisor: if C is the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$, and C' the pull-back of the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$, then $D_{I,J}$ is the zero locus of the natural map $s_{i_0}^* \Omega_C \to s_{i_0}^* \Omega_{C'}$ of the pull-backs of the relative cotangent sheaves for some $i_0 \in I$. By the universal property of blow-ups, this shows that $\rho_{\mathcal{B},\mathcal{A}}$ factors via the blow-up $\rho' \colon M \to \overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$ of $\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$ at C_{IJ}.

We now construct the inverse map. Let C' be the pull-back of the universal curve along ρ', let E be the exceptional divisor of ρ', and write $\rho'^{-1} s_i \colon M \to C'$ for the pull-back of the sections s_i over $\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$. Let C_0 be the common image $(\rho'^{-1} s_i)(E)$ of the exceptional divisor for any $i \in I$, and let C be blow-up of C' at C_0. The center $C_0 \subset C'$ is a codimension two regular embedding, and embeds as a Cartier divisor in both $(\rho'^{-1} s_i)(M)$ for any $i \in I$, and in the restriction of C' to E. Thus the fibers of C over E are obtained from that of the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$ by attaching a projective line at the marked point given by any s_i for $i \in I$, and every section $\rho'^{-1} s_i$ lifts to a section $s_i \colon M \to C$ via the proper transform of $(\rho'^{-1} s_i)(M)$.

Over E, the image is contained in the attached projective line, away from the node, as $s_i(M)$ and the fiber over E meet transversely in C'. Also, since the images of $s_i, i \in I$ intersect transversely in the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{B}}(V,\beta)$, any tangent vector at a point of C_0 tangent to all the images of $(\rho'^{-1} s_i)(M), i \in I$ is already tangent to C_0; thus the sections $s_i \colon M \to C$ cannot all be mapped to the same point of the projective line.

Hence, with the induced map to V, we have constructed a $(g,\mathcal{A})$-stable map, and so a map $M \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$; it is an inverse to the map in the opposite direction constructed above, as this is true over $C_{g,S}(V,\beta)$ and both stacks are separated.

3.1.3. **Proposition.** *Let $\mathcal{A}, \mathcal{B}$ as in proposition 3.1.1, except we allow some weights of $\mathcal{B}$ to be zero. Let $i \in S$ be a label with $\mathcal{A}(i) > \mathcal{B}(i) = 0$. Then $\rho_{\mathcal{A},\mathcal{B}}$ additionally*

contracts the boundary components $C_{(g_1,0,g_2),(I_1,I_0,I_2),(\beta_1,0,\beta_2)}$ *which are defined as the image of the gluing morphisms*

$$\overline{\mathcal{M}}_{g_1,\mathcal{A}|_{I_1}\cup\{1\}}(V,\beta_1)\times_V\overline{\mathcal{M}}_{0,\mathcal{A}_{I_0\cup\{i\}}\cup\{1,1\}}(V,0)\times_V\overline{\mathcal{M}}_{g_2,\mathcal{A}|_{I_2}\cup\{1\}}(V,\beta_2) \to \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$$

for all $g_1+g_2=g$, $\beta_1+\beta_2=\beta$ *and disjoint partitions* $I_1\cup I_0\cup\{i\}\cup I_2=S$ *such that* $\mathcal{A}(j)=0$ *for* $j\in I_0$.

The restriction $\rho_{\mathcal{B},\mathcal{A}}$ *factors via the projection of the second component to a point.*

In other words, this is the boundary component of singular curves such that the section s_i is contained in a node after stabilization.

3.2. **Chamber decomposition.** We now assume $\beta\neq 0$, and consider the set of positive weights $\mathcal{D}_n=(0,1]^S\subset\mathbb{R}^S$. The walls $\mathcal{W}_c$ and $\mathcal{W}_f$ of the coarse and fine chamber decomposition, respectively, are given by:[2]

$$\mathcal{W}_c=\left\{\sum\nolimits_{i\in I}\mathcal{A}(i)=1\,\middle|\, I\subset S, 2<|I|\right\}$$
$$\mathcal{W}_f=\left\{\sum\nolimits_{i\in I}\mathcal{A}(i)=1\,\middle|\, I\subset S, 2\le|I|\right\}$$

Coarse and fine chambers are connected component of the complements $\mathcal{D}_n\setminus\mathcal{W}_c$ and $\mathcal{D}_n\setminus\mathcal{W}_f$, respectively.

3.2.1. **Proposition.** *(cf.* [Has03, Proposition 5.1]*) The coarse chamber decomposition is the coarsest decomposition such that* $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ *is constant in each chamber. The fine chamber decomposition is the coarsest decomposition such that the universal curve* $\mathcal{C}_{g,\mathcal{A}}(V,\beta)$ *is constant in each chamber.*

3.2.2. **Corollary.** *Let* $\mathcal{A}$ *be positive weight data in the interior of a fine open chamber. Then for small* $\epsilon>0$*, the forgetful morphism* $\phi_{\mathcal{A},\mathcal{A}\cup\{\epsilon\}}$ *identifies* $\overline{\mathcal{M}}_{g,\mathcal{A}\cup\{\epsilon\}}(V,\beta)$ *with the universal curve* $C_{g,\mathcal{A}}(V,\beta)\to\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$.

This holds by definition for $\epsilon=0$, and it follows easily from the earlier propositions that $\rho_{\mathcal{A}\cup\{0\},\mathcal{A}\cup\{\epsilon\}}$ is an isomorphism.

§4. Virtual fundamental classes and Gromov-Witten invariants

From now on, we assume additionally that the target V is smooth.

4.1. **Expected properties.** The crucial step in the construction of Gromov-Witten invariants is the construction of virtual fundamental classes of expected dimension:

$$[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}\in A_{(1-g)(\dim V-3)-K_V\cdot\beta+|S|}\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$$

We will provide now a basic list of properties that such a construction should satisfy, and proceed to draw some conclusions about Gromov-Witten invariants in the remainder of the section.

(1) *Mapping to a point.* If $\beta=0$, then

$$[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,0)]^{\mathrm{virt}}=c_{g\dim V}(R^1\pi_*f^*TV)$$

[2] The conditions $|S|<n-2$ and $|S|\le n-2$ for the coarse and fine chamber decompositions, respectively, in [Has03, section 5] are correct only when $g=0$ and don't apply in our case as we assumed $\beta\neq 0$.

(2) *Forgetting a tail.* Assume $\mathcal{A}$ and ϵ are as in corollary 3.2.2, so that $\phi_{\mathcal{A},\mathcal{A}\cup\epsilon}$ is the universal curve over $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$. In particular, this implies that $\phi_{\mathcal{A},\mathcal{A}\cup\{\epsilon\}}$ is flat, and thus defines a pull-back in intersection theory. We require

$$\phi_{\mathcal{A},\mathcal{A}\cup\epsilon}(V,\beta)^*[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,\mathcal{A}\cup\epsilon}(V,\beta)]^{\mathrm{virt}}.$$

(3) *Combining tails.* Assume we are in the situation of proposition 2.4. Since all sections lie in the smooth locus of the curve, $\mu_{S/S'}$ is a regular embedding, and we require that

$$\mu^!_{S/S'}[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,\mathcal{A}|_{S'}\cup\{\mathcal{A}(S'')\}}(V,\beta)]^{\mathrm{virt}}.$$

(4) *Gluing.* We fix $g_1, S_1, \mathcal{A}_1, g_2, S_2, \mathcal{A}_2$ and some $\beta \in H_2^+(V)$. Set $g = g_1 + g_2$ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Consider the gluing morphisms

$$\mu_{\beta_1,\beta_2} \colon \overline{\mathcal{M}}_{g_1,\mathcal{A}_1\cup\{1\}}(V,\beta_1) \times \overline{\mathcal{M}}_{g_2,\mathcal{A}_2\cup\{1\}}(V,\beta_2) \times_{V\times V} V$$
$$\to \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$$

of proposition 2.1.1 for all β_1, β_2 with $\beta_1 + \beta_2 = \beta$. The union of their images is the boundary component in $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$ given as the pull-back

$$\begin{tikzcd} \overline{\mathcal{M}}_{(g_1,\mathcal{A}_1)|(g_2,\mathcal{A}_2)}(V,\beta) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta) \arrow[d] \\ \overline{\mathcal{M}}_{g_1,\mathcal{A}_1\cup\{1\}} \times \overline{\mathcal{M}}_{g_2,\mathcal{A}_2\cup\{1\}} \arrow[r, "\mu"] & \overline{\mathcal{M}}_{g,\mathcal{A}} \end{tikzcd}$$

Since the moduli spaces of weighted stable curves are smooth, μ is a l.c.i. morphism and defines a pull-back $\mu^![\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}$. On the other hand, via the diagonal $\Delta\colon V \to V \times V$, we can pull-back the virtual fundamental class on the product $\overline{\mathcal{M}}_{g_1,\mathcal{A}_1\cup\{1\}}(V,\beta_1) \times \overline{\mathcal{M}}_{g_2,\mathcal{A}_2\cup\{1\}}(V,\beta_2)$ to the fiber product that is the source of μ_{β_1,β_2}. We require

$$\sum_{\beta_1+\beta_2=\beta} \mu_{\beta_1,\beta_1 *}\Delta^! \left([\overline{\mathcal{M}}_{g_1,\mathcal{A}_1\cup\{1\}}(V,\beta_1)]^{\mathrm{virt}} \times [\overline{\mathcal{M}}_{g_2,\mathcal{A}_2\cup\{1\}}(V,\beta_2)]^{\mathrm{virt}}\right)$$
$$= \mu^![\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}.$$

(5) *Kontsevich-stable maps.* If all weights are 1, then $[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}$ agrees with the definition of virtual fundamental classes of [BF97, Beh97].

(6) *Reducing weights.* Given two set of weights $\mathcal{A} > \mathcal{B}$, we require compatibility with the reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$:

$$\rho_{\mathcal{B},\mathcal{A}*}[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}$$

Evidently, properties (1), (2) and (4) are direct generalizations of properties satisfied by the virtual fundamental classes of the non-weighted moduli spaces, while (3) and (6) are new.

4.1.1. **Theorem.** *There is a system of virtual fundamental classes satisfying all of the above properties.*

While the Behrend-Fantechi construction can be applied to our situation and provides virtual fundamental classes, we instead use (5) and (6) as a definition, and prove that these classes automatically satisfy the desired properties.

We postpone the proof of the above properties to §6, after having generalized them to a bigger class of morphisms labelled by a category of weighted stable graphs. In the remainder of the section we will instead proceed to give some consequences of theorem 4.1.1.

4.2. **Gromov-Witten invariants.** As in the non-weighted case, one defines the n-point Gromov-Witten invariants of V depending on weights $\mathcal{A}\colon \{1,\dots,n\} \to [0,1] \cap \mathbb{Q}$ via

$$\langle\ \rangle_{g,\mathcal{A},\beta}\colon H^*(V)^{\otimes n} \to \mathbb{C}$$

$$\langle \gamma_1 \otimes \cdots \otimes \gamma_n \rangle_{g,\mathcal{A},\beta} = \int_{[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

and Gromov-Witten invariants including gravitational descendants via

$$\langle \tau_1^{k_1}\gamma_1 \cdots \tau_n^{k_n}\gamma_n \rangle_{g,\mathcal{A},\beta} = \int_{[\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)]^{\mathrm{virt}}} \psi_1^{k_1}\mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \psi_n^{k_n}\mathrm{ev}_n^*(\gamma_n)$$

where ψ_i is the tautological class associated to the section s_i: $\psi_i = c_1(s_i^*\Omega_C)$ where Ω_C is the relative cotangent bundle of the universal curve C over $\overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta)$.

4.2.1. **Proposition.** *Gromov-Witten invariants without gravitational descendants do not depend on the choice of weights $\mathcal{A}$.*

It is enough to prove this for two weights $\mathcal{A} > \mathcal{B}$. The evaluation morphisms $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,\mathcal{A}}(V,\beta) \to V$ factor via the reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$. Hence the claim follows from property (6) and the projection formula.

4.3. **Extended modular operad.** Let $\mathcal{A}_{m,n}$ be the weight data consisting of m weights of one, and n weights of $\epsilon \leq \frac{1}{n}$. The moduli spaces $\overline{\mathcal{M}}_{g,\mathcal{A}_{m,n}}$ were called $L_{g,m,n}$ in [LM04] and studied more closely in [Man04]. Markings with weight one and ϵ are white and black points in the language of [LM04], respectively: white points may not coincide with any other point, whereas any number of black points are allowed to coincide. Similarly, we write $L_{g,m,n}(V,\beta)$ for the moduli spaces of weighted stable maps $L_{g,m,n}(V,\beta) = \overline{\mathcal{M}}_{g,\mathcal{A}_{m,n}}(V,\beta)$.

In [LM04], the notion of an $\mathcal{L}$-algebra was introduced by a combinatorial description. It is an extension of the graph-level description of the genus zero-part of a cohomological field theory in the sense of [KM94]. Equivalently, it can be given by a system of cohomology classes in $L_{0,m,n}$ (rather than classes in $L_{0,m,0} \cong \overline{\mathcal{M}}_{0,m}$). An $\mathcal{L}$-algebra yields a (formal) solution to the commutativity equations, which are extension of the WDVV equations.

By the results of [Man04], the "economy class description" of an $\mathcal{L}$-algebra give in [LM04, section 4.2.2] can be translated into the following cohomological description:

Let $(T; F, (,))$ be a triple consisting of two $\mathbb{Z}_2$-graded $\mathbb{Q}$-vector spaces T, F, where the latter is equipped with a (super)symmetric non-degenerate scalar product $(,)$. An $\mathcal{L}$-algebra on $(T; F, (,))$ over a $\mathbb{Q}$-algebra R can be given as a collection of maps

$$I_{0;m,n}\colon T^{\otimes n} \otimes F^{\otimes m} \to H_*(L_{0;m,n}) \otimes_{\mathbb{Q}} R$$

being compatible with gluing of black points and the trace on F.

We obtain the *$\mathcal{L}$-algebra of quantum cohomology of V including gravitational descendants* as follows: Let $F = H^*(V, \mathbb{Q})$, equipped with the Poincaré pairing, and let $T = \bigoplus_{k\geq 0} z^k F$. We denote by $\mathrm{ev}_1^W, \dots, \mathrm{ev}_m^W$ and $\mathrm{ev}_1^B, \dots, \mathrm{ev}_n^B$ the evaluation maps $L_{0;m,n}(V,\beta) \to V$ induced by the marked sections of weight one and ϵ, respectively, and by $\pi\colon L_{0;m,n}(V,\beta) \to L_{0;m,n}$ the forgetful map. Let $\psi_i, i = 1\dots n$ be the tautological classes associated to the section s_i^B of weight ϵ. Let $\mathbb{Q}[[q]]$ be the Novikov ring of V, i.e. the formal completion of the polynomial ring over the semigroup of effective classes in $H_2(V)/\text{torsion}$.

Then we define $I_{0;m,n}$ as

$$I_{0;m,n}\left(z^{k_1}\gamma_1 \otimes \dots \otimes z^{k_n}\gamma_n \otimes \delta_1 \otimes \dots \otimes \delta_m\right)$$
$$= \sum_{\beta \in H_2^+(V)} q^\beta \mathcal{P}\left(\pi_*\left(\prod_{i=1}^n (\mathrm{ev}_i^B)^*\gamma_i\psi_i^{k_i}\prod_{i=1}^n(\mathrm{ev}_i^W)^*\delta_i \cap [L_{0,m,n}(V,\beta)]^{\mathrm{virt}}\right)\right)$$

where $\pi\colon L_{0;m,n}(V,\beta) \to L_{0;m,n}$ is the forgetful map, and $\mathcal{P}(s) \in H^*L_{0;m,n}$ is the Poincaré dual of $s \in H_*L_{0;m,n}$.

4.3.1. **Theorem.** *The above definition of $I_{0;m,n}$ yields a cyclic $\mathcal{L}$-algebra (in the sense of the economy class description in* [LM04, section 4.2.2]*).*

The only thing to check is the compatibility with gluing, in the formal sense of [LM04, diagram (4.8)]. This holds due to property (4) of section 4.1.

4.4. **Comments.** In [LM04], it was shown that the datum of an $\mathcal{L}$-algebra is equivalent to a geometric structure, a solution of the so-called commutativity equation. However, the structure of an $\mathcal{L}$-algebra does not capture the complete structure we have available:

(1) By property (6), the inclusion $F = z^0F \subset T$ is compatible with the reduction morphisms $L_{0,m,n} \to L_{0,m-1,n+1}$ in the obvious sense.
(2) Relating the gravitational descendants to intersection numbers in $L_{0;m,n}$ by an analysis analogous to the one in [KM98] will, of course, lead to many more relations among the correlators.

One might hope that these can be integrated in the geometric picture of [LM04].

As a side remark, it is worth pointing out that the tautological classes $\psi_i, i = 1\dots n$ in $L_{0;m,n}(V,\beta)$ are compatible with pull-back along the forgetful morphism $L_{0;m,n+1}(V,\beta)$; this is not true in the non-weighted case.

§5. **Graph-language**

5.1. **Weighted marked graphs.** The elementary morphisms described in §2 generate a larger system of morphisms. They are best modelled over a category of weighted marked graphs; this category generalizes the category of marked graphs introduced in [BM96] by introducing weights of tails. We follow [BM96, section 1] closely.

We recall from section 2.5 the definition of a graph:

5.1.1. **Definition.** [BM96, Definition 1.1] *A graph τ is a quadruple $(F_\tau, V_\tau, j_\tau, \partial_\tau)$ of a finite set V_τ of vertices, a finite set F_τ of flags, an involution $j_\tau\colon F_\tau \to F_\tau$*

and a map $\partial_\tau\colon F_\tau \to V_\tau$. *We call* $S_\tau = \{f \in F_\tau | j_\tau f = f\}$ *the set of tails, and* $E_\tau = \{\{f, j_\tau f\} | f \in F_\tau$ *and* $j_\tau f \neq f\}$ *the set of edges.*

5.1.2. **Definition.** *A weighted modular graph is a graph* $\tau = (F_\tau, V_\tau, j_\tau, \partial_\tau)$ *endowed with two maps* $g_\tau\colon V_\tau \to \mathbb{Z}_{\geq 0}$ *and* $\mathcal{A}_\tau\colon F_\tau \to \mathbb{Q} \cap (0, 1]$ *such that* $\mathcal{A}_\tau(f) = 1$ *for all flags* f *that are part of an edge, i.e. for which* $j_\tau(f) \neq f$.

The number $g_\tau(v)$ is called the genus of a vertex, and $\mathcal{A}_\tau(f)$ the weight of a flag.

5.1.3. **Definition.** *Given a semigroup* A *with indecomposable zero, a weighted* A*-graph* (τ, α) *is a weighted modular graph* τ *with a map* $\alpha\colon V_\tau \to A$. *A weighted marked graph is a pair* $(A, (\tau, \alpha))$ *where* A *is a semigroup with indecomposable zero, and* (τ, α) *is an* A*-graph.*

We will often omit α from the notation and call τ an A-graph.

Morphisms in the category of weighted marked graphs are generated by two different types, *combinatorial morphisms* and *contractions*. More precisely, since the associated geometric morphisms are contravariant with respect to the combinatorial morphisms, and covariant with respect to contractions, the morphisms will be generated by contractions and formal inverses of the combinatorial morphisms.

Only condition (2) of the definition of a combinatorial morphism of modular graphs ([BM96, Definition 1.7]) needs to be adapted to our situation:

5.1.4. **Definition.** *Let* (σ, α) *and* (τ, β) *be weighted* A*-graphs. A combinatorial morphism* $a\colon (\sigma, \alpha) \to (\tau, \beta)$ *is a pair of maps* $a_F\colon F_\sigma \to F_\tau$ *and* $a_V\colon V_\sigma \to V_\tau$*, satisfying the following conditions:*

(1) *The morphisms commute with* ∂*, i.e. we have* $a_V \circ \partial_\sigma = \partial_\tau \circ a_F$. *In particular, for any* $v \in V_\sigma$ *and* $w = a_V(v) \in V_\tau$*, we get an induced map* $a_{V,v}\colon F_\sigma(v) \to F_\tau(w)$.
(2) *Consider the above map* $a_{V,v}$. *Then for any* $f \in F_\tau(w)$*, the inequality*
$$\sum_{f' \in F_\sigma(v)\colon\, a_{V,v}(f') = f} \mathcal{A}_\sigma(f') \leq \mathcal{A}_\tau(f)$$
is satisfied.
(3) *Let* $\{f, \bar{f}\}$ *be an edge of* σ*, i.e.* $f \in F_\sigma, \bar{f} = j_\sigma(f) \neq f$. *Then there exist* $n \geq 1$ *and* n *edges* $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$ *of* τ *such that* $v_i := \partial_\tau(\bar{f}_i) = \partial_\tau(f_{i+1})$ *and* $\beta(v_i) = 0$ *for all* $1 \leq i < n$.
(4) *For every* $v \in V_\sigma$ *we have* $\alpha(v) = \beta(a_V(v))$.
(5) *For every* $v \in V_\sigma$ *we have* $g(v) = g(a_V(v))$.

A combinatorial morphism of weighted marked graphs $(B, \sigma, \beta) \to (A, \tau, \alpha)$ *is a pair* (ξ, a) *where* $\xi\colon A \to B$ *is a homomorphism of semigroups, and* $a\colon (\sigma, \beta) \to (\tau, \xi \circ \alpha)$ *is a combinatorial morphism of* B*-graphs.*

Note that we do not require that j_σ and j_τ commute with a_F and a_V; in particular, σ could be obtained from τ by cutting an edge into two tails. Other examples of combinatorial morphisms are morphisms adding tails or adding connected components. There are essentially two new types of morphisms compared to the non-weighted case:

(1) (*Combining tails.*) Consider a subset $\{t_1, \dots, t_n\} \in F_\sigma(v)$ of tails attached to a vertex v, and assume that its sum of weights satisfies $\sum_i \mathcal{A}_\sigma(t_i) \leq 1$. Then

we can form a new graph τ by replacing the tails $\{t_1, \ldots, t_n\}$ with a single tail $\bar{t}$ of weight $\mathcal{A}_\tau(\bar{t}) := \sum_i \mathcal{A}_\sigma(t_i)$.

(2) (*Increasing the weights.*) This means that (τ, β) are identical to (σ, α) as modular graphs, but the weight data $\mathcal{A}_\tau$ satisfies $\mathcal{A}_\tau \geq \mathcal{A}_\sigma$.

We refer to [BM96, Definition 1.3] for the definition of a contraction $\phi\colon \tau \to \sigma$ of graphs. It is obtained by collapsing a subgraph consisting entirely of edges (and the adjoining vertices) to one vertex for every connected component of the subgraph. It is given by an injective map $\phi^F: F_\sigma \to F_\tau$ (which is bijective on tails) and a surjective map $\phi_V: V_\tau \to V_\sigma$.

5.1.5. **Definition.** *A contraction of weighted marked graphs $\phi\colon (\tau, \beta) \to (\sigma, \alpha)$ is a contraction $\phi\colon \tau \to \sigma$ of graphs such that*

(1) $\alpha(v) = \sum_{w \in \phi_V^{-1}(v)} \alpha(w)$ *for all $v \in V_\sigma$,*
(2) $g(v) = \sum_{w \in \phi_V^{-1}(v)} \alpha(w) + H^1(|\tau_v|)$ *for all $v \in V_\sigma$ and τ_v being the subgraph of τ being collapsed onto v, and*
(3) $\mathcal{A}_\tau(\phi^F(f)) = \mathcal{A}_\sigma(f)$ *for all tails $f \in S_\sigma$.*

5.1.6. **Definition.** *A vertex v of a weighted modular A-graph (τ, α) is called stable if $\alpha(v) \neq 0$ or $2g(v) - 2 + \sum_{f \in F_\tau\colon \partial_\tau(f) = v} \mathcal{A}_\tau(f) > 0$. A graph is stable if all its vertices are stable.*

5.1.7. **Remark.** *Let (τ, α) be a weighted A-graph. There is a unique weighted stable A-graph (τ^s, α^s) and a combinatorial morphism $(\tau^s, \alpha^s) \to (\tau, \alpha)$, such that every combinatorial morphism $(\sigma, \beta) \to (\tau, \alpha)$ from a stable A-graph (σ, β) factors uniquely through (τ^s, α^s).*

The graph (τ^s, α^s) is called the *stabilization* of (τ, α). Similarly, there is a stabilization of weighted modular graphs. The stabilization τ^s of the underlying modular graph τ of an A-graph (τ, α) is also called the *absolute stabilization*.

The stabilization (τ^s, α^s) can be constructed via a sequence of steps as below, following [BM96, Proposition 1.13]:

(1) If there is a connected component of τ that has only one vertex, and this vertex is unstable, we remove this connected component from τ.
(2) If there is an unstable vertex v attached to one edge $\{f_0, \bar{f}_0 = j_\tau(f_0)\}$ with $\partial_\tau(f_0) = v$, $\partial_\tau(\bar{f}_0) \neq v$ and $n \geq 0$ tails $f_1, \ldots, f_n$, we remove the vertex v and the flags $f_0, \ldots, f_n$ from the graph and modify j such that $j(\bar{f}_0) = \bar{f}_0$, i.e. the edge becomes a tail at the vertex $\partial_\tau(\bar{f}_0)$ with weight one.
(3) If there is an unstable vertex v attached to two edges $\{f_1, \bar{f}_1 = j_\tau(f_1)\}$ and $\{f_2, \bar{f}_2 = j_\tau(f_2)\}$ with $\partial_\tau(f_i) = v$ and $\partial_\tau(\bar{f}_i) \neq v$, we remove v and the tails f_i from the graph, and modify j such that $j(\bar{f}_1) = \bar{f}_2$. In other words, we combine the tails $\bar{f}_1, \bar{f}_2$ to form a new edge.

At every step, any combinatorial morphism $(\sigma, \beta) \to (\tau, \alpha)$, where (σ, β) is a stable V-graph, factors uniquely through the new graph, and the claim of the remark follows by induction on the number of unstable vertices.

5.1.8. **Definition.** *Let (A, τ) and (B, σ) be weighted stable marked graphs. A morphism $(A, \tau) \to (B, \sigma)$ is quadruple (ξ, a, τ', ϕ) where $\xi\colon A \to B$ is a homomorphism*

of semigroups, τ' is a weighted stable B-graph, $a\colon \tau' \to \tau$ makes (ξ, a) into a combinatorial morphism of weighted marked graphs, and $\phi\colon \tau' \to \sigma$ is a contraction of B-graphs.

$$\begin{tikzcd} B & \tau' \arrow[r, "\phi"] \arrow[d, "a"] & \sigma \\ A \arrow[u, "\xi"] & \tau & \end{tikzcd}$$

We think of this morphism as the composition of ϕ with the inverse of (ξ, a), except that (ξ, a) itself is not a morphism in the category of weighted stable marked graphs. As explained earlier, this construction is motivated by the fact that the geometric morphisms are covariant with respect to contractions, but contravariant with respect to combinatorial morphisms.

To define compositions, we need the definition of stable pullback; the construction of [BM96] applies with minor changes. Given a combinatorial morphism of weighted marked graphs $(a, \xi)\colon (B, \rho) \to (A, \tau)$ and a contraction of weighted A-graphs $\phi\colon \sigma \to \tau$, it canonically constructs a weighted stable B-graph π, together with a contraction of B-graphs $\psi\colon \pi \to \rho$ and a combinatorial morphism of weighted marked graphs $b\colon \pi \to \sigma$:

$$\begin{tikzcd} B & \pi \arrow[r, "\psi"] \arrow[d, "b"] & \rho \arrow[d, "a"] \\ A \arrow[u, "\xi"] & \sigma \arrow[r, "\phi"] & \tau \end{tikzcd}$$

We call π the *stable pullback of ρ under ϕ*. We will describe how to obtain π from ρ, assuming that ϕ is an elementary contraction (i.e. it contracts a single edge).

If ϕ contracts a loop adjacent to a vertex $v \in V_\tau$, we simply reattach a loop at every preimage $v' \in a_V^{-1}(v)$ (and decrease its genus by one). If ϕ contracts an edge $\{f, \bar{f}\}$ connecting the vertices $v_1 = \partial_\sigma(f), v_2 = \partial_\sigma(\bar{f})$, let $v = \phi_V(v_1) = \phi_V(v_2)$ their common image in τ, and let $v' \in a_V^{-1}$ be any vertex in the preimage of v in ρ. There can be two cases:

(1) Replace v' by two vertices v_1', v_2' connected by an edge $\{f', \bar{f}'\}$; their class and genus are determined by the corresponding vertex in σ: $\alpha_\pi(v_i') = \xi(\alpha_\sigma(v_i))$ and $g_\pi(v_i') = g_\sigma(v_i)$. A flag f_1 of v is moved to v_1' or v_2' according to its position in σ, i.e. according to whether $\phi^F(a_F(f_1))$ is attached to v_1 or v_2; its weight remains unchanged. Now if either v_1' or v_2' is unstable, we undo this construction and skip to case (2). Otherwise, it remains to define the maps: ψ is the map contracting $\{f', \bar{f}'\}$; the combinatorial morphism b is given by sending v_i' to v_i, and by sending a flag $f_1 \neq f'$ of v_i' to $\left(\phi^F \circ a \circ (\psi^F)^{-1}\right)(f_1)$. Other than that, b agrees with a.

(2) Assume in the above construction, the vertex v_2' was unstable. We leave ρ unchanged, and let b_V send v' to v_1. Let f_1 be a flag of v'; we set $b_F(f_1) = \phi^F(a_V(f_1))$ if that is a flag attached to v_1, otherwise $b_F(f_1) = f$, where f defined above is part of the edge connection v_1 and v_2.

The same construction is iteratively applied to every such vertex v to obtain π.

Geometrically, the contractions ϕ corresponds to the inclusion of a boundary component $\overline{\mathcal{M}}(\sigma)$ of the moduli space $\overline{\mathcal{M}}(\tau)$ associated to τ, and the stable pull-back constructs the boundary component of $\overline{\mathcal{M}}(\rho)$ upon which the boundary component $\overline{\mathcal{M}}(\sigma)$ is naturally mapped by morphism $\overline{\mathcal{M}}(\tau) \to \overline{\mathcal{M}}(\rho)$ associated to a.

5.1.9. **Proposition and Definition.** *Let $(\xi, a, \tau', \phi) \colon (A, \tau) \to (B, \sigma)$ and $(\eta, b, \sigma', \psi) \colon (B, \sigma) \to (C, \rho)$ be morphisms of weighted stable marked graphs. Then we define the composition $(\eta, b, \sigma', \psi) \circ (\xi, a, \tau', \phi) \colon (A, \tau) \to (C, \rho)$ to be $(\eta\xi, ac, \tau'', \psi\xi)$ where (c, τ'', ξ) is the stable pullback of σ' under ϕ.*

This composition is associative, defining the category of weighted stable marked graphs.

$$\begin{tikzcd}
C & & \tau'' \arrow[r, "\xi"] \arrow[d, "c"] & \sigma' \arrow[r, "\psi"] \arrow[d, "b"] & \rho \\
B \arrow[u, "\eta"] & & \tau' \arrow[r, "\phi"] \arrow[d, "a"] & \sigma & \\
A \arrow[u, "\xi"] & & \tau & &
\end{tikzcd}$$

We denote by $\mathfrak{G}_s^w$ the category of weighted stable marked graphs, and by $\mathfrak{A}$ the category of semigroups with indecomposable zeros.

5.2. **Weighted stable maps indexed by graphs.** As in [BM96, section 3], let $\mathfrak{V}$ be the category of smooth projective varieties over a field k. Consider the fibered product $\mathfrak{V}\mathfrak{G}_s^w$ of categories

$$\begin{tikzcd}
\mathfrak{V}\mathfrak{G}_s^w \arrow[r] \arrow[d] & \mathfrak{G}_s^w \arrow[d] \\
\mathfrak{V} \arrow[r, "H_2^+"] & \mathfrak{A}
\end{tikzcd}$$

where H_2^+ is the functor that associates to V the semigroup of effective classes in $\mathrm{CH}^1(V)$. Objects of $\mathfrak{V}\mathfrak{G}_s^w$ are pairs (V, τ) where V is a smooth projective variety over k and τ is a weighted stable $H_2^+(V)$-graph.

For any weighted graph τ and any vertex $v \in V_\tau$, let $F_v = \{f \in F_\tau | \partial_\tau(f) = v\}$ be the set of flags attached to v, and $\mathcal{A}_v = \mathcal{A}|_{F_v}$ be their weight data.

5.2.1. **Definition.** *A stable map of type (V, τ) for an object (V, τ) in $\mathfrak{V}\mathfrak{G}_w^w$ is a collection of stable maps (C_v, x_v, f_v) to V of type $(g(v), \mathcal{A}_v, \alpha(v))$ for every $v \in V_\tau$, such that $f_{\partial_\tau(i)}(x_i) = f_{\partial_\tau(j_\tau(i))}(x_{j_\tau(i)})$ for all flags i.*

For a scheme T and $(V, \tau) \in \mathfrak{V}\mathfrak{G}_s^w$, let $\overline{\mathcal{M}}(T)(V, \tau)$ be the groupoid of families of weighted stable maps of type (V, τ) over T, and let $\overline{\mathcal{M}}(T)$ be the groupoid of arbitrary weighted stable maps.

5.2.2. **Theorem.** *For a fixed scheme T, $\overline{\mathcal{M}}(T)$ defines a 2-functor*

$$\overline{\mathcal{M}}(T)(\_\_) \colon \mathfrak{V}\mathfrak{G}_s^w \to \overline{\mathcal{M}}(T).$$

For every base change $u \colon T' \to T$, the pullback $u^* \colon \overline{\mathcal{M}}(T) \to \overline{\mathcal{M}}(T')$ commutes with the functors $\overline{\mathcal{M}}(T)(\_\_)$ and $\overline{\mathcal{M}}(T')(\_\_)$.

Finally, for fixed (V, τ, α)*, the category of weighted stable maps of type* (V, τ, α) *is a proper algebraic Deligne-Mumford stack* $\overline{\mathcal{M}}(V, \tau, \alpha)$ *of finite type.*

Of course, the compatibility with base change in particular implies that that $\overline{\mathcal{M}}(\Phi)$ for some morphism Φ in $\mathfrak{V}\mathfrak{G}_s^w$ induces a morphisms between the stacks associated by $\overline{\mathcal{M}}$ to the source and target; i.e. $\overline{\mathcal{M}}$ is a 2-functor from $\mathfrak{V}\mathfrak{G}_s^2$ to the 2-category of Deligne-Mumford stacks.[3]

The last claim of the theorem immediately follows from theorem 1.1.4 and the fact that by definition it is a closed substack of $\prod_{v \in V_\tau} \overline{\mathcal{M}}_{g(v), \mathcal{A}(v)}(V, \alpha(v))$.

To prove the first and second claim of the theorem, we need to prove the existence of a functorial push-forward in $\overline{\mathcal{M}}(T)$ associated to every morphism $(\xi, a, \tau', \phi) \colon (V, \tau) \to (W, \sigma)$ in $\mathfrak{V}\mathfrak{G}_s^w$, and show that they are compatible with base change. Every morphism in $\mathfrak{V}\mathfrak{G}_s^w$ can be written as a composition of elementary morphisms of one of the following types: changing the target (I), increasing the weights (II), forgetting a tail (III), combining tails (IV), complete combinatorial morphisms (V), contracting an edge (VI) and contracting a loop (VII). For complete combinatorial morphisms this is immediate (and there is nothing to add to the discussion in [BM96, section 3, case IV]). All other cases have already been treated in §2 in the case where the target is a one-vertex graph; the general case follows immediately from this.

What is left to prove is that the associated morphism are compatible with composition in the category of weighted stable marked graphs, i.e. that it does not depend on the way we break up a morphism into a composition of elementary morphisms.

For compositions of contractions with contractions, respectively of the (inverses of) combinatorial morphisms with combinatorial morphisms this is immediate, and the only interesting case to prove is the case of the composition $(\xi, a)^{-1} \circ \phi$ of (the formal inverse of) a combinatorial morphism $(\xi, a) \colon (B, \rho) \to (A, \tau)$ and a contraction of A-graphs $\phi \colon \sigma \to \tau$. In fact, the formation of stable pull-back exactly makes sure that this compatibility holds, and the claim follows easily by following every step of the stable pull-back construction.

§6. **Graph-level description of virtual fundamental classes.**

To define Gromov-Witten invariants based on weighted stable maps, we need to define virtual fundamental classes in the Chow ring $A_*\left(\overline{\mathcal{M}}(V, \tau, \alpha)\right)$ of the moduli spaces. To formulate the required behavior with respect to restriction to boundary components of the moduli space, we need to introduce the notion of isogenies of weighted stable graphs and their cartesian isogeny diagrams. (We won't introduce the complete cartesian extended isogeny category as in [BM96].)

6.1. **Isogenies of graphs.** For our purposes, we need to refine the definition of an isogeny as given in [BM96, Definition 5.4].

6.1.1. **Definition.** *We say that the one-vertex* V*-graph* σ *is a* contraction of small tails *of the one-vertex* V*-graph* τ *if it is obtained from* τ *by a sequence of steps, each forgetting a single tail, such that in every step we are in the situation of corollary 3.2.2 (the weight data of* τ *is contained in a fine open chamber, and the weight of the*

[3] Implicitly, we passed from the description of a stack as a category fibered in groupoids to the description as a 2-functor to the 2-category of groupoids. See e.g. [Man99, Chapter V] for a discussion of both viewpoints.

additional tail in σ is small enough that changing it to zero would not cross a wall of the fine chamber decomposition).

This implies that the associated map $\overline{\mathcal{M}}(\tau) \to \overline{\mathcal{M}}(\sigma)$ is flat, as it is a sequence of projection maps of the universal curve.

6.1.2. **Definition.** *An isogeny $\Phi\colon \tau \to \sigma$ of weighted stable A-graphs is given by an injective map $\Phi^F : F_\sigma \to F_\tau$ of flags and a surjective map $\Phi_V\colon V_\tau \to V_\sigma$ of vertices such that the following conditions hold:*

(1) *Φ^F commutes with the boundary maps $\partial_\tau, \partial_\sigma$, i. e. for any flag $f \in F_\sigma$, we have $\Phi_V(\partial_\tau(\Phi^F(f))) = \partial_\sigma(f)$.*
(2) *For any vertex $v \in V_\sigma$, let τ_v be the subgraph of τ that consists of all vertices send to v by Φ_V, and all edges joining them. We require that*
  (a) $g(v) = \sum_{w\in V_{\tau_v}} g(w) + \dim H^1(|\tau_v|)$ *and*
  (b) $\alpha(v) = \sum_{w\in V_{\tau_v}} \alpha(w)$
(3) *Φ^F respects the weights, i.e. $\mathcal{A}_\tau \circ \Phi^F = \mathcal{A}_\sigma$.*
(4) *For any $v \in V_\tau$, let τ_v be the one-vertex graph obtained from τ by removing all other vertices, and cutting off the edges starting from v into a tail of weight 1; let σ_v be the graph obtained from τ_v by removing all tails not in the image of Φ^F. The condition is that σ_v is is a contraction of small tails of τ_v.*

In the geometric realizations of the graphs, an isogeny is given by collapsing a set of disjoint closed connected subgraphs $|\tau_v| \subset |\tau|$ consisting of edges and small tails to a single vertex $v \in V_\sigma$. It can be written as the composition of a morphism contracting small tails, and a contraction as in definition 5.1.5.

6.2. **Cartesian isogeny diagrams.** Consider a stable V-graph σ and its absolute stabilization $a\colon \sigma^s \to \sigma$, as well as an isogeny of weighted modular graphs $\Phi\colon \tau^s \to \sigma^s$. In [BM96, section 5] the pull-back $\tau = (\tau_i)_{i\in I}$ of σ along Φ is constructed. For each $i \in I$, the stable V-graph τ_i comes with a stabilization morphism $a_i\colon \tau^s \to \tau_i$ and an isogeny $\Phi_i\colon \tau_i \to \sigma$ such that the diagram

$$\begin{tikzcd} \tau_i \arrow[r, "\Phi_i"] & \sigma \\ \tau^s \arrow[u, "a_i"] \arrow[r, "\Phi"] & \sigma^s \arrow[u, "b"'] \end{tikzcd}$$

commutes.

Its construction is as follows:[4] To every edge $\{f, \bar f\}$ of σ^s there is a long edge in σ consisting of edges $\{f_1, \bar f_1\}, \dots, \{f_n, \bar f_n\}$ and vertices $v_i = \partial_\sigma(\bar f_i) = \partial_\sigma(f_{i+1})$ such that $b^F(f) = f_1$, $b^F(\bar f) = \bar f_n$ and the vertices v_i are of genus 0 and have no further flags. We replace the edge $\{\Phi^F(f), \Phi^F(\bar f)\}$ of τ^s by the same long edge $\{f_1, \bar f_1\}, \dots, \{f_n, \bar f_n\}$. Similarly, to every tail $f \in S_{\sigma^s}$ there is a long tail $\{f_1, \bar f_1\}, \dots, \{f_n, \bar f_n\}$ of edges as above and some number $k \geq 0$ of additional tails $f_{n+1}, \dots, f_{n+k}$. The additional tails are attached to the last vertex v_n of the tail, $\partial_\sigma(f_{n+i}) = v_n = \partial_\sigma(\bar f_n)$ for $1 \leq i \leq k$, and the sum of weights is bounded as $\sum_{1\leq i\leq k} \mathcal{A}(f_{n+i}) \leq 1$. Again we replace the tail $\Phi^F(f) \in S_{\sigma^s}$ with the same long tail, preserving the weights.

[4]Unlike [BM96, section 5], we omit the orbit map as well as the notion of an extended isogeny.

We thus obtain a weighted graph τ' with a combinatorial morphism $a\colon \tau^s \to \tau'$ and an isogeny of graphs $\Phi'\colon \tau' \to \sigma$. Now let I be the set of V-structures on τ' such that Φ' is an isogeny of weighted V-graphs. We get a set $(\tau_i)_{i\in I}$ of V-graphs such that the induced morphism $a_i\colon \tau^s \to \tau_i$ is an absolute stabilization, and $\Phi_i\colon \tau_i \to \sigma$ is an isogeny of V-graphs.

The same construction can be made for a tuple $(\sigma_j)_{j\in J}$ of V-graphs with absolute stabilization morphisms $b_j\colon \sigma \to \sigma_j$. The formation of pull-back then becomes compatible with composition.

6.3. **Expected properties.**

6.3.1. **Definition.** *Let τ be a weighted stable V-graph, where V is of pure dimension $\dim V$, and has canonical class ω_V. We define the class $\beta(\tau)$, the Euler characteristic $\chi(\tau)$, the genus $g(\tau)$ and the dimension $\dim(\tau)$ of τ as*

$$\begin{aligned}
\beta(\tau) &= \sum_{v\in V_\tau} \beta(v)\\
\chi(\tau) &= \chi(|\tau|) - \sum_{v\in V_\tau} g(v)\\
g(\tau) &= 1 - \chi(\tau)\\
\dim(\tau) &= \chi(\tau)(\dim V - 3) - \beta(\tau)\cdot\omega_V + |S_\tau| - |E_\tau|
\end{aligned}$$

We now fix V. An orientation will be a system of virtual fundamental classes $J(V,\tau) \subset A_{\dim(V,\tau)}(\overline{\mathcal{M}}(V,\tau))$ for all stable V-graphs τ bounded by the characteristic, satisfying the list of properties given below.

(1) (*Mapping to a point*). If τ is a graph of class zero, and $|\tau|$ is nonempty and connected, then

$$J(V,\tau) = c_{g(\tau)\dim V}\left(R^1\pi_* f^* TV\right).$$

(2) (*Forgetting tails*). Let $\Phi\colon \sigma \to \tau$ be a morphism of stable V-graphs given by forgetting a small tail of σ, i.e. such that τ is obtained from σ by a contraction of a small tail. Then $M(\Phi)$ is flat, and we require

$$J(V,\sigma) = \overline{\mathcal{M}}(\Phi)^* J(V,\tau).$$

(3) (*Combining tails.*). Let $\Phi\colon \sigma \to \tau$ be a morphism splitting up a tail into several of them, i. e. one that is induced by a combinatorial morphism $a\colon \tau \to \sigma$ combining several tails $f_1,\dots,f_k \in S_\tau$ to a single tail $f \in S_\sigma$ with weight $\mathcal{A}_\sigma(f) = \sum_{i=1}^k \mathcal{A}_\tau(f_i)$. Then $\overline{\mathcal{M}}(\Phi)$ is a regular closed embedding, and the required condition is

$$J(V,\sigma) = \overline{\mathcal{M}}(\Phi)^! J(V,\tau).$$

(4a) (*Products*). For any two stable V-graphs σ,τ, let $\sigma\times\tau$ be the disjoint union of the graphs of σ and τ with the obvious structure as a stable V-graph. Then

$$J(V,\sigma\times\tau) = J(V,\sigma)\times J(V,\tau).$$

(4b) (*Cutting edges*). Let $\Phi\colon \sigma \to \tau$ be a morphism obtained by cutting an edge $\{f, \bar{f}\}$ of σ into two tails. By abuse of notation, we identify the flags $f, \bar{f} \subset F_\sigma$ with the corresponding tails $f, \bar{f} \subset S_\tau$. We obtain a cartesian square

$$\begin{tikzcd} \overline{\mathcal{M}}(V,\sigma) \arrow[r, "\overline{\mathcal{M}}(\Phi)"] \arrow[d, "\mathrm{ev}_f = \mathrm{ev}_{\bar{f}}"] & \overline{\mathcal{M}}(V,\tau) \arrow[d, "\mathrm{ev}_f \times \mathrm{ev}_{\bar{f}}"] \\ V \arrow[r, "\Delta"] & V \times V \end{tikzcd}$$

and require that

$$J(V,\sigma) = \Delta^! J(V,\tau).$$

(4c) (*Isogenies*). Let $(\sigma_j)_{j\in J}$ be a tuple of V-graphs with absolute stabilization σ^s and $\tau^s \to \sigma^s$ an isogeny. Let $(\tau_i)_{i\in I}$ be the tuple of V-graphs completing this to a cartesian isogeny diagram. We obtain an induced commutative, but not cartesian diagram

$$\begin{tikzcd} \coprod_{i\in I} \overline{\mathcal{M}}(\tau_i) \arrow[r] \arrow[d] & \coprod_{j\in J} \overline{\mathcal{M}}(\sigma_j) \arrow[d] \\ \overline{\mathcal{M}}(\tau^s) \arrow[r, "\overline{\mathcal{M}}(\Phi)"] & \overline{\mathcal{M}}(\sigma^s) \end{tikzcd}$$

and thus an induced map

$$h\colon \coprod_{i\in I} \overline{\mathcal{M}}(\tau_i) \to \overline{\mathcal{M}}(\tau^s) \times_{\overline{\mathcal{M}}(\sigma^s)} \coprod_{j\in J} \overline{\mathcal{M}}(\sigma_j).$$

We require that

$$h_*\left(\sum_{i\in I} J(V,\tau_i)\right) = \sum_{j\in J} \overline{\mathcal{M}}(\Phi)^! J(V,\sigma_j).$$

(5) *Kontsevich-stable maps.* Assume that all weights satisfy $\mathcal{A}(s) = 1$. Then $J(V,\tau)$ agrees with the definition of the virtual fundamental class $J(V,\tilde{\tau})$ for the underlying stable V-graphs $\tilde{\tau}$ according to [Beh97, BF97].

(6) *Reducing weights.* Let $\Phi\colon \sigma \to \tau$ be a morphism of weighted stable V-graphs obtained by reducing weights, i. e. such that Φ is induced by a combinatorial morphism $\tau \to \sigma$ that is the identity on the modular graph structure, but such that $\mathcal{A}_\sigma(f) \geq \mathcal{A}_\tau(f)$ for all flags $f \in F_\tau = F_\sigma$. Then $\overline{\mathcal{M}}(\Phi)$ is a reduction morphism, and we require that

$$\overline{\mathcal{M}}(\Phi)_*\left(J(V,\sigma)\right) = \left(J(V,\tau)\right).$$

6.3.2. **Theorem.** *There is a system of virtual fundamental classes satisfying all properties listed in the previous section.*

Note that (4a), (4b) and (4c) imply condition (4) of theorem 4.1.1, whereas the other conditions for one-vertex graphs are identical to the corresponding condition in [BM96].

Of course, (1), (2) and (4a-c) are direct generalizations of properties of the virtual fundamental classes in the non-weighted setting. The only caveat is that for morphisms contracting or forgetting a tail, we always have to assume the situation of corollary

3.2.2. This is to be expected: if we forget a tail of bigger weight, the forgetful map factorizes via a non-trivial reduction morphism ρ. However, there is no reason to assume that the virtual fundamental class is a pull-back of a class via ρ.

As we already explained, we use (5) and (6) as the definition:

6.3.3. **Definition and Remark.** *For any weighted stable V-graph τ, let τ^1 be the weighted stable V-graph obtained by setting all weights to 1, let $w(\tau)\colon \tau \to \tau^1$ be the combinatorial morphism increasing the weights, and $W(\tau)\colon \tau^1 \to \tau$ the induced morphism in the category of weighted marked graphs. Then any combinatorial morphism $\tau \to \sigma$ to a V-graph σ with all weights equal to 1 factors uniquely via $w(\tau)$.*

By abuse of notation, we write $W(\tau)\colon \overline{\mathcal{M}}(V,\tau^1) \to \overline{\mathcal{M}}(V,\tau)$ also for the induced map on moduli spaces, and define $J(V,\tau)$ as

$$J(V,\tau) := W(\tau)_* J(V,\tau^1)$$

where the latter is as defined in [Beh97, BF97].

We will now show how to obtain these properties from those listed in Definition 7.1 in [BM96], which have been verified for the Behrend-Fantechi construction of the virtual fundamental class in [Beh97]. As a preparation, we need the following lemma:

6.3.4. **Lemma.** *Let $\Phi\colon \sigma \to \tau$ be an isogeny of V-graphs, and let $\Phi^1\colon \sigma^1 \to \tau^1$ be the same morphism for the graphs with weight 1. Consider the commutative (but not necessarily cartesian) square*

$$\begin{array}{ccc}
\overline{\mathcal{M}}(V,\sigma^1) & \xrightarrow{\overline{\mathcal{M}}(\Phi^1)} & \overline{\mathcal{M}}(V,\tau^1) \\
\Big\downarrow{\scriptstyle \overline{\mathcal{M}}(W(\sigma))} & & \Big\downarrow{\scriptstyle \overline{\mathcal{M}}(W(\tau))} \\
\overline{\mathcal{M}}(V,\sigma) & \xrightarrow[\overline{\mathcal{M}}(\Phi)]{} & \overline{\mathcal{M}}(V,\tau)
\end{array}$$

and the induced morphism $h\colon \overline{\mathcal{M}}(V,\sigma^1) \to \overline{\mathcal{M}}(V,\sigma) \times_{\overline{\mathcal{M}}(V,\tau)} \overline{\mathcal{M}}(V,\tau^1)$. Then $\overline{\mathcal{M}}(\Phi)^!$ and $h_* \circ \overline{\mathcal{M}}(\Phi^1)^!$ yield the same orientation to the projection

$$\overline{\mathcal{M}}(V,\sigma) \times_{\overline{\mathcal{M}}(V,\tau)} \overline{\mathcal{M}}(V,\tau^1) \to \overline{\mathcal{M}}(V,\tau^1).$$

(By definition, an orientation of a morphism $f\colon X \to Y$ is an element of the bivariant intersection theory $A^*(Y \to X)$, i.e. in particular a morphism $A_*(X') \to A_*(Y')$ for every pull-back $f'\colon X' \to Y'$ of f.)

We may assume that Φ is an elementary isogeny, so we have one of the following two cases:

- *Contraction of an edge*. It is sufficient to consider the case where τ has only one vertex, so both $\overline{\mathcal{M}}(\Phi)$ and $\overline{\mathcal{M}}(\Phi^1)$ are a gluing morphism as in proposition 2.1.1. Consider the first case, where Φ contracts a non-looping edge (the other case follows similarly). An object in the product consists of a pair of weighted stable maps $((C_1,f_1),(C_2,f_2))$ of type σ and τ^1, respectively, together with an isomorphism the reduction of C_2 to type τ with the curve obtained by gluing the two components of C_1. Since the sections cannot meet the node, this is only possible if C_2 already consists of two components, which together form a weighted stable maps of type σ^1. The induced map to $\overline{\mathcal{M}}(V,\sigma^1)$ is an inverse to h, i.e. the above diagram is a cartesian square.

Both $\overline{\mathcal{M}}(\Phi)$ and $\overline{\mathcal{M}}(\Phi^1)$ are a codimension one regular embedding with compatible normal bundle, and the claim follows by standard intersection theory.

- *Contraction of a small tail.* In this case, both $\overline{\mathcal{M}}(\Phi)$ and $\overline{\mathcal{M}}(\Phi^1)$ are flat. The orientation given by $\overline{\mathcal{M}}(\Phi)$ is the same as that of the projection to the second factor of the product. Since h is a blow-up at a regularly embedded substack, we have $h_* \circ h^* = \mathrm{id}$, and the assertion follows.

We proceed with the proof of theorem 6.3.2.

(1) This follows from the same property [BM96, Definition 7.1, (1)] in the non-weighted case and projection formula.

(2) Consider the diagram of lemma 6.3.4:

$$\begin{aligned}
\overline{\mathcal{M}}(\Phi)^! J(V,\tau) &= \overline{\mathcal{M}}(\Phi)^! \overline{\mathcal{M}}(W(\tau))_* J(V,\tau^1) && \text{(by definition)} \\
&= p_{1*}\overline{\mathcal{M}}(\Phi)^! J(V,\tau^1) && \text{(push-forward)} \\
&= p_{1*}h_*\overline{\mathcal{M}}(\Phi^1)^! J(V,\tau^1) && \text{(by lemma 6.3.4)} \\
&= \overline{\mathcal{M}}(W(\sigma))_* J(V,\sigma^1) && (*) \\
&= J(V,\sigma) && \text{(by definition)}
\end{aligned}$$

Here (*) holds by [BM96, Definition 7.1, (4)].

(4a) This is obvious from the same property for non-weighted graphs [BM96, Definition 7.1, (2)].

(4b) The natural map $\overline{\mathcal{M}}(V,\sigma^1) \to \overline{\mathcal{M}}(V,\tau^1)$ fits as an additional row on top of diagram given in condition (4b), so that all squares are cartesian. Thus the claim follows from property [BM96, Definition 7.1, (3)] and push-forward.

(4c) We may assume that $|J| = 1$, so we are just dealing with a single V-graph σ and its absolute stabilization σ^s.

Consider σ^1 and its absolute stabilization $(\sigma^1)^s$. By the universal property of stabilization, the composition of the combinatorial morphisms of weighted graphs $\sigma^s \to \sigma \to \sigma^1$ factors uniquely via $(\sigma^1)^s$. Similarly, for each $i \in I$ let τ_i^1 be the corresponding graphs with weights 1, and let, by some abuse of notation, $(\tau^1)^s$ be their common absolute stabilization; we obtain a combinatorial morphism $\tau^s \to (\tau^1)^s$.

These morphisms can be completed to the following diagram of a cube:

$$\begin{tikzcd}
 & \coprod_i \tau_i \arrow[rr, "\coprod \Phi_i"] \arrow[dd] & & \sigma \arrow[dd] \\
\coprod_i \tau_i^1 \arrow[ur, "W(\tau_i)"] \arrow[rr, "\coprod \Phi_i^1"] \arrow[dd] & & \sigma^1 \arrow[ur, "W(\sigma)"'] \arrow[dd] & \\
 & \tau^s \arrow[rr, "\Phi"] & & \sigma^s \\
(\tau^1)^s \arrow[ur] \arrow[rr, "\Phi^1"] & & (\sigma^1)^s \arrow[ur] &
\end{tikzcd}$$

More precisely, there exist unique contractions $\Phi^1\colon (\tau^1)^s \to (\sigma^1)^s$ and $\Phi_i^1\colon \tau_i^1 \to \sigma^1$ such that

(I) the top and bottom square are commutative in the category of weighted marked graphs, and

(II) the square in front is a cartesian isogeny diagram.

Assuming these claims, the desired property can be deduced from the corresponding property [BM96, Definition 7.1, (5)] by careful diagram computation:

Since none of the squares of the cube necessarily yield cartesian squares of moduli spaces, we need to consider the products $P_{\mathrm{back}} = \overline{\mathcal{M}}(\tau^s) \times_{\overline{\mathcal{M}}(\sigma^s)} \overline{\mathcal{M}}(V,\sigma)$, $P_{\mathrm{front}} = \overline{\mathcal{M}}((\tau^1)^s) \times_{\overline{\mathcal{M}}((\sigma^1)^s)} \overline{\mathcal{M}}(V,\sigma^1)$ and $P_{\mathrm{diag}} = \overline{\mathcal{M}}(\tau^s) \times_{\overline{\mathcal{M}}(\sigma^s)} \overline{\mathcal{M}}(V,\sigma^1)$. Let h_{back} and h_{front} be the induced map from the corresponding corner of the cube to P_{back} and P_{front}, respectively, and $h_{\mathrm{d}\to\mathrm{b}} \colon P_{\mathrm{diag}} \to P_{\mathrm{back}}$, $h_{\mathrm{f}\to\mathrm{b}} \colon P_{\mathrm{front}} \to P_{\mathrm{back}}$ and $h_{\mathrm{f}\to\mathrm{d}} \colon P_{\mathrm{front}} \to P_{\mathrm{diag}}$ the maps induced by the commutative cube. We obtain

$$\begin{aligned}
\overline{\mathcal{M}}(\Phi)^! J(V,\sigma) &= \overline{\mathcal{M}}(\Phi)^! \overline{\mathcal{M}}(W(\sigma))_* J(V,\sigma^1) && \text{(by definition)}\\
&= h_{\mathrm{d}\to\mathrm{b}*} \overline{\mathcal{M}}(\Phi)^! J(V,\sigma^1) && \text{(push-forward)}\\
&= h_{\mathrm{d}\to\mathrm{b}*} h_{\mathrm{f}\to\mathrm{d}*} \overline{\mathcal{M}}(\Phi^1)^! J(V,\sigma^1) && \text{(lemma 6.3.4)}\\
&= h_{\mathrm{f}\to\mathrm{b}*} h_{\mathrm{front}*} \sum_i J(V,\tau_i^1) && \text{(*)}\\
&= h_{\mathrm{back}*} \sum_i W(\tau_i)_* J(V,\tau_i^1) && \\
&= h_{\mathrm{back}*} \sum_i J(V,\tau_i), && \text{(by definition)}
\end{aligned}$$

where (*) holds according to [BM96, Definition 7.1, (5)]. So it remains to prove the two claims above.

The definition of Φ_i^1 is obvious and necessarily unique, as the graphs τ_i and τ_i^1, as well as σ_i and σ_i^1, are identical as marked graphs after forgetting the weights. Commutativity of the top square is equivalent to the claim that the combinatorial morphism $w(\tau_i) \colon \tau_i \to \tau_i^1$ is the stable pull-back (see p. 18) of $w(\sigma) \colon \sigma \to \sigma^1$ along Φ_i^1, which is equally obvious.

For the bottom square involving Φ^1, we need to review the construction of cartesian isogenies. Consider any tail $f \in S_{\sigma^s}$; it corresponds to a long tail in σ consisting of edges $\{f_1, \bar{f}_1\}, \dots, \{f_n, \bar{f}_n\}$, of vertices $v_1, \dots, v_n$ and of tails $f_{n+1}, \dots, f_{n+k}$ attached to v_n. Its preimage $\Phi^F(f) \in S_{\tau^s}$ corresponds to an identical long tail $\{\Phi_i^F(f_1), \Phi_i^F(\bar{f}_1)\}$, ... etc. in τ_i. After adjusting the weights to one, we again see identical long tails as part of σ^1 respectively τ_i^1; these will have identical stabilization in $(\sigma^1)^s$ resp. $(\tau^1)^s$. This shows that Φ^1 is uniquely determined on the stabilization of this long tail. The same discussion applies to any edge of σ^s corresponding to a long edge in σ^s. Finally, any part of τ^s contracted by Φ will appear identically in τ_i, and thus in τ_i^1 and $(\tau^1)^s$. Hence Φ^1 will necessarily contract it, too.

We have thus constructed Φ^1 so that the front square is a cartesian isogeny diagram. At the same time, the above discussion shows that the stable pull-back of $\sigma^s \to (\sigma^1)^s$ along Φ^1 will recover $\tau^s \to (\tau^1)^s$, i.e. the bottom square is indeed commutative.

(5) This holds by definition.

(6) This follows from the definition and the fact that reduction morphisms are compatible with composition (Proposition 1.2.1).

(3) By properties (4a) and (4b), we can consider only graphs having a single vertex. Further, we may assume that the combinatorial morphism a combines exactly two tails $f_1, f_2 \in S_\tau$ to a single tail $f = a_F(f_1) = a_F(f_2) \in S_\sigma$.

Let ρ be the V-graph obtained from σ^1 by adding a second vertex of class and genus zero, having two tails f'_1, f'_2 of weight 1 and one edge whose second flag connects it to the original vertex and replaces the tail f; geometrically, we attach a tripod[5] to the tail f.

The morphism $\rho \to \sigma^1$ induced by the combinatorial morphism $\sigma^1 \to \rho$ gives an isomorphism of moduli spaces $\overline{\mathcal{M}}(\rho) \to \overline{\mathcal{M}}(\sigma^1)$, which respects the virtual fundamental classes by properties (1), (4a) and (4b).

There is a morphism $\Psi\colon \rho \to \tau^1$ contracting the edge in ρ and sending f'_i to f_i. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}(\rho) \cong \overline{\mathcal{M}}(\sigma^1) \arrow[r, "\overline{\mathcal{M}}(\Psi)"] \arrow[d, "W(\sigma)"] & \overline{\mathcal{M}}(\tau^1) \arrow[d, "W(\tau)"] \\
\overline{\mathcal{M}}(\sigma) \arrow[r, "\overline{\mathcal{M}}(\Phi)"] & \overline{\mathcal{M}}(\tau)
\end{tikzcd}$$

A discussion similar to the one in the proof of (4c) shows that this is a cartesian square. Let $\Xi\colon \tau^1 \to \sigma^1$ be the morphism obtained by forgetting the tail f_1 and mapping f_2 to f. Then $\overline{\mathcal{M}}(\Psi)$ is a section of $\overline{\mathcal{M}}(\Xi)$, so $\overline{\mathcal{M}}(\Psi)^![\overline{\mathcal{M}}(\tau^1)]^{\mathrm{virt}} = \overline{\mathcal{M}}(\Psi)^!\overline{\mathcal{M}}(\Xi)^*[\overline{\mathcal{M}}(\sigma^1)]^{\mathrm{virt}} = [\overline{\mathcal{M}}(\sigma^1)]^{\mathrm{virt}}$. The desired equality follows by push-forward and the vanishing of excess intersection.

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